

A.1 Topological spaces and basic topological notions

Definition. **Topological space** is a pair (X, \mathcal{T}) , where X is a set and \mathcal{T} is a family of subsets of X , satisfying the following properties:

- (a) $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$.
- (b) If $\mathcal{A} \subset \mathcal{T}$ is any subfamily, then $\bigcup \mathcal{A} \in \mathcal{T}$.
- (c) For any two sets $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

A family \mathcal{T} with these properties is called a **topology** on X . Instead of (X, \mathcal{T}) we often write just X (if we know which topology is considered).

Definition. Let (X, \mathcal{T}) be a topological space.

- A set $A \subset X$ is said to be **open in** (X, \mathcal{T}) (or **\mathcal{T} -open**, or just **open**), if $A \in \mathcal{T}$.
- Let $A \subset X$ and $x \in A$. The point x is said to be an **interior point** of the set A if there is an open set B such that $x \in B \subset A$.
- **The interior** of a set $A \subset X$ is the set of all its interior points. The interior of A is denoted $\text{Int } A$ or, more precisely, $\text{Int}_{\mathcal{T}} A$.
- A set $A \subset X$ is said to be a **neighborhood of the point** $x \in X$ if x is an interior point of A .
- Let $A \subset X$ and $x \in X$. The point x is said to be a **boundary point** of the set A if for each neighborhood U of x we have $U \cap A \neq \emptyset$ and simultaneously $U \cap (X \setminus A) \neq \emptyset$.
- **The boundary** of a set $A \subset X$ is the set of all its boundary points. The boundary of A is denoted ∂A or, more precisely, $\partial_{\mathcal{T}} A$. (Sometimes the boundary of A is denoted $H(A)$ or $\text{bd } A$.)
- A set $A \subset X$ is said to be **closed**, if it contains all its boundary points, i.e. if $\partial A \subset A$.
- **The closure** of a set $A \subset X$ is the set $A \cup \partial A$. The closure of A is denoted \bar{A} or, more precisely $\bar{A}^{\mathcal{T}}$. (Sometimes the closure of A is denoted by $\text{cl } A$ or $\text{cl}_{\mathcal{T}} A$ or $\mathcal{T} - \text{cl } A$.)

Proposition 1. Let (X, \mathcal{T}) be a topological space and $A \subset X$.

- (i) The interior of A is the largest open set contained in A .
- (ii) The set A is closed if and only if $X \setminus A$ is open.
- (iii) The closure of A is the smallest closed set containing A .
- (iv) Let $x \in X$. Then $x \in \bar{A}$ if and only if for each neighborhood U of x we have $U \cap A \neq \emptyset$.

Proposition 2 (properties of open sets). Let (X, \mathcal{T}) be a topological space.

- (a) \emptyset and X are closed sets.
- (b) If \mathcal{A} is any family of closed subsets of X , then $\bigcap \mathcal{A}$ is closed as well.
- (c) For any two closed sets $C, D \subset X$ the set $C \cup D$ is closed.

Definition. Let (X, \mathcal{T}) be a topological space and $\mathcal{B} \subset \mathcal{T}$.

- The family \mathcal{B} is said to be a **base** (or **basis**) of the topology \mathcal{T} if for any $U \in \mathcal{T}$ and any $x \in U$ there exists $G \in \mathcal{B}$ such that $x \in G \subset U$.
- The family \mathcal{B} is said to be a **subbase** (or **subbasis**) of the topology \mathcal{T} if for any $U \in \mathcal{T}$ and any $x \in U$ there exist $G_1, \dots, G_k \in \mathcal{B}$ such that $x \in G_1 \cap \dots \cap G_k \subset U$.

Remark. Let (X, \mathcal{T}) be a topological space and $\mathcal{B} \subset \mathcal{T}$.

- \mathcal{B} is a base of \mathcal{T} if and only if for each $U \in \mathcal{T}$ there is $\mathcal{A} \subset \mathcal{B}$ with $\bigcup \mathcal{A} = U$.
- \mathcal{B} is a subbase of \mathcal{T} if and only if the family of all the sets which can be expressed as the intersection of finitely many elements of \mathcal{B} forms a base of \mathcal{T} .

Proposition 3. Let X be a set and let \mathcal{B} be a family of subsets of X .

- (i) The family \mathcal{B} is a base of some topology on X if and only if the following two conditions are fulfilled:
- $\bigcup \mathcal{B} = X$;
 - For any $U, V \in \mathcal{B}$ and any $x \in U \cap V$ there exists $W \in \mathcal{B}$ with $x \in W \subset U \cap V$.
- (ii) The family \mathcal{B} is a subbase of some topology on X if and only if $\bigcup \mathcal{B} = X$.

Definition. Let (X, \mathcal{T}) be a topological space, $a \in X$ and \mathcal{U} be a family of subsets of X . The family \mathcal{U} is said to be a **base of neighborhoods of the point a** if the following two conditions hold:

- Each $U \in \mathcal{U}$ is a neighborhood of a .
- For any neighborhood V of a there is $U \in \mathcal{U}$ with $U \subset V$.

Proposition 4. Let X be a set and, for each $x \in X$, let \mathcal{U}_x be a family of subsets of X . Then there is a topology \mathcal{T} on X such that for each $x \in X$ the family \mathcal{U}_x is a base of neighborhoods of x , if and only if the following conditions are fulfilled:

- (a) $x \in U$ whenever $x \in X$ and $U \in \mathcal{U}_x$.
- (b) If $x \in X$ and $U, V \in \mathcal{U}_x$ then there is $W \in \mathcal{U}_x$ such that $W \subset U \cap V$.
- (c) For any $x \in X$ and any $U \in \mathcal{U}_x$ there is $V \subset X$ such that $x \in V \subset U$ and, moreover,

$$\forall y \in V \exists W \in \mathcal{U}_y : W \subset V.$$

The topology \mathcal{T} is then uniquely determined and

$$\mathcal{T} = \{U \subset X; \forall x \in U \exists V \in \mathcal{U}_x : V \subset U\}.$$

Example. Let (X, ρ) be a metric space.

- For $x \in X$ and $r > 0$ we set $U(x, r) = \{y \in X; \rho(x, y) < r\}$. Then

$$\mathcal{T} = \{U \subset X; \forall x \in U \exists r > 0 : U(x, r) \subset U\}$$

is a topology on X . It is the **topology generated by the metric ρ** .

- Let $x \in X$. Any of the following families is a base of neighborhoods of x :

$$\{U(x, r); r > 0\}; \quad \{U(x, \frac{1}{n}); n \in \mathbb{N}\}; \quad \{\overline{U(x, \frac{1}{n})}; n \in \mathbb{N}\}.$$

A.2 Continuous mappings

Definition. Let (X, \mathcal{T}) and (Y, \mathcal{U}) be topological spaces and let $f : X \rightarrow Y$ be a mapping.

- (1) The mapping f is said to be **continuous at $x \in X$** if for each neighborhood V of $f(x)$ in (Y, \mathcal{U}) there exists a neighborhood U of x in (X, \mathcal{T}) such that $f(U) \subset V$.
- (2) The mapping f is said to be **continuous on X** if it is continuous at each $x \in X$.

Proposition 5 (characterizations of continuity). Let (X, \mathcal{T}) and (Y, \mathcal{U}) be topological spaces and let $f : X \rightarrow Y$ be a mapping. The following assertions are equivalent:

- (i) f is continuous on X .
- (ii) For any open set $U \subset Y$ the set $f^{-1}(U)$ is open in X .
- (iii) For any closed set $F \subset Y$ the set $f^{-1}(F)$ is closed in X .
- (iv) For any set $A \subset X$ we have $f(\overline{A}) \subset \overline{f(A)}$.

A.3 Separation axioms

Definition. Let (X, \mathcal{T}) be a topological space. The space X is said to be

- T_0 , if for any two distinct points $a, b \in X$ there exists $U \in \mathcal{T}$ containing exactly one of the points a, b ;
- T_1 , if for any two distinct points $a, b \in X$ there exists $U \in \mathcal{T}$ such that $a \in U$ and $b \notin U$;
- T_2 (or **Hausdorff**), if for any two distinct points $a, b \in X$ there exist $U, V \in \mathcal{T}$ such that $a \in U, b \in V$ and $U \cap V = \emptyset$;
- **regular**, if for any $a \in X$ and any closed set $B \subset X$ with $a \notin B$ there exist $U, V \in \mathcal{T}$ such that $a \in U, B \subset V$ and $U \cap V = \emptyset$;
- T_3 , if it is T_1 and regular;
- **completely regular**, if for any $a \in X$ and any closed set $B \subset X$ with $a \notin B$ there exists a continuous function $f : X \rightarrow \mathbb{R}$ such that $f(a) = 1$ and $f|_B = 0$;
- $T_{3\frac{1}{2}}$ (or **Tychonoff**), if it is T_1 and completely regular;
- **normal**, if for any two disjoint closed sets $A, B \subset X$ there exist $U, V \in \mathcal{T}$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$;
- T_4 , if it is T_1 and normal.

Remark.

- Trivially $T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$.
- $T_4 \Rightarrow T_{3\frac{1}{2}}$ holds as well, but it is not trivial, it is a consequence of the Urysohn lemma.
- Any metric space is T_4 .

Proposition 6 (Urysohn lemma). *Let X be a normal topological space and $A, B \subset X$ two disjoint closed sets. Then there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.*

A.4 Subspaces, products and quotients

Definition. Let (X, \mathcal{T}) be a topological space and $Y \subset X$. Then $\mathcal{T}_Y = \{U \cap Y; U \in \mathcal{T}\}$ is a topology on Y and the space (Y, \mathcal{T}_Y) is then a **topological subspace** of the space (X, \mathcal{T}) .

Remark. Any subspace of a $T_0, T_1, T_2, \text{regular}, T_3, \text{completely regular}$ or $T_{3\frac{1}{2}}$ space enjoys the same property. (This is obvious.) A subspace of a T_4 space need not be T_4 . (This is not obvious.)

Definition. Let $(X_1, \mathcal{T}_1), \dots, (X_k, \mathcal{T}_k)$ be nonempty topological spaces. By their **cartesian product** we mean the set $X_1 \times \dots \times X_k$ equipped with the topology, whose base is

$$\{U_1 \times \dots \times U_k; U_1 \in \mathcal{T}_1, \dots, U_k \in \mathcal{T}_k\}.$$

Definition. Let $(X_\alpha, \mathcal{T}_\alpha), \alpha \in A$, be any nonempty family of nonempty topological spaces. By their **cartesian product** we mean the set $\prod_{\alpha \in A} X_\alpha$ equipped with the topology, whose base is

$$\left\{ \left\{ f \in \prod_{\alpha \in A} X_\alpha; f(\alpha_1) \in U_1, \dots, f(\alpha_k) \in U_k \right\}; \right. \\ \left. U_1 \in \mathcal{T}_{\alpha_1}, \dots, U_k \in \mathcal{T}_{\alpha_k}, \alpha_1, \dots, \alpha_k \in A, k \in \mathbb{N} \right\}$$

Proposition 7. Let $(X_\alpha, \mathcal{T}_\alpha)$, $\alpha \in A$, be any nonempty family of nonempty topological spaces and let $\prod_{\alpha \in A} X_\alpha$ be their cartesian product. Let (Y, \mathcal{U}) be a topological space and $f : Y \rightarrow \prod_{\alpha \in A} X_\alpha$ a mapping. The mapping f is continuous on Y if and only if for each $\alpha \in A$ the mapping $y \mapsto f(y)(\alpha)$ is a continuous mapping of Y to X_α .

Definition. Let (X, \mathcal{T}) be a topological space, Y a set and $f : X \rightarrow Y$ an onto mapping. The **quotient topology** on Y induced by the mapping f is the topology

$$\mathcal{T}_Y = \{U \subset Y; f^{-1}(U) \in \mathcal{T}\}.$$

Definition. Let (X, \mathcal{T}) and (Y, \mathcal{U}) be topological spaces and $f : X \rightarrow Y$ an onto mapping. We say that f is a **quotient mapping** if \mathcal{U} is the quotient topology induced by the mapping f .

Proposition 8. Let (X, \mathcal{T}) and (Y, \mathcal{U}) be topological spaces and $f : X \rightarrow Y$ a continuous onto mapping. If f is open (i.e., $f(U)$ is open in Y for each open $U \subset X$) or closed (i.e., $f(F)$ is closed in Y for each closed $F \subset X$), then f is a quotient mapping.

Proposition 9. Let (X, \mathcal{T}) and (Y, \mathcal{U}) be topological spaces and $f : X \rightarrow Y$ a quotient mapping. Let (Z, \mathcal{W}) be a topological space and let $g : Y \rightarrow Z$ be a mapping. Then g is continuous if and only if $g \circ f$ is continuous.

A.5 Compact spaces

Definition. A topological space (X, \mathcal{T}) is said to be **compact**, if for any family \mathcal{U} of open sets covering X (i.e. satisfying $\bigcup \mathcal{U} = X$) there exists a finite subfamily $\mathcal{W} \subset \mathcal{U}$ covering X (i.e. such that $\bigcup \mathcal{W} = X$.)

Proposition 10. Let X be a compact topological space and $Y \subset X$ its topological subspace.

- If Y is closed in X , then Y is compact.
- If X is Hausdorff and Y is compact, then Y is closed in X .

Proposition 11. Let X be a compact topological space, Y a topological space and $f : X \rightarrow Y$ a continuous onto mapping. Then:

- (i) Y is compact.
- (ii) If Y is Hausdorff, then f is a closed mapping (and hence a quotient mapping).
- (iii) If Y is Hausdorff and f is one-to-one, then f is a homeomorphism (i.e., f^{-1} is continuous as well).

Proposition 12. Any Hausdorff compact topological space is T_4 , and hence also $T_{3\frac{1}{2}}$.

Theorem 13 (Tychonoff theorem). The cartesian product of any family of Hausdorff compact topological spaces is compact. In particular, the spaces $[-1, 1]^\Gamma$, $[0, 1]^\Gamma$, $\{0, 1\}^\Gamma$ and $\{z \in \mathbb{C}; |z| \leq 1\}^\Gamma$ are compact for any set Γ .

A.6 Convergence of sequences and nets

Definition. Let X be a topological space, (x_n) a sequence of elements of X and $x \in X$. We say that the sequence (x_n) **converges to x in the space X** , if for any neighborhood U of x there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ we have $x_n \in U$. The point x is then called a **limit of the sequence** (x_n) , we write $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$.

Remark: If X is Hausdorff, then each sequence has at most one limit.

Proposition 14. *Let X be a metric space. Then:*

(1) *Let $A \subset X$. Then*

$$\bar{A} = \{x \in X; \exists (x_n) \text{ a sequence in } A : x_n \rightarrow x\}$$

(2) *Let $A \subset X$. Then A is closed if and only if any $x \in X$, which is the limit of a sequence in A , belongs to A .*

(3) *Let Y be a topological space, $f : X \rightarrow Y$ a mapping and $x \in X$. The mapping f is continuous at x , if and only if*

$$\forall (x_n) \text{ sequence in } X : x_n \rightarrow x \Rightarrow f_n(x) \rightarrow f(x).$$

Definition. Let (Γ, \preceq) be a partially ordered set. We say that it is **directed** (more precisely **up-directed**), if for any pair $\gamma_1, \gamma_2 \in \Gamma$ there exists $\gamma \in \Gamma$ such that $\gamma_1 \preceq \gamma$ and $\gamma_2 \preceq \gamma$.

Examples of directed sets:

- $\Gamma =$ the set of all finite subsets of \mathbb{N} , $A \preceq B \equiv^{\text{df}} A \subset B$.
- $\Gamma =$ the set of all neighborhoods of x in a topological space X , $U \preceq V \equiv^{\text{df}} U \supset V$.

Definition. Let X be a topological space and let (Γ, \preceq) be a directed set.

- By a **net indexed by Γ** we mean any mapping $\alpha : \Gamma \rightarrow X$.
- We say that a net $\alpha : \Gamma \rightarrow X$ **converges to $x \in X$** if

$$\forall U \text{ neighborhood of } x \exists \gamma_0 \in \Gamma \forall \gamma \in \Gamma, \gamma \succeq \gamma_0 : \alpha(\gamma) \in U.$$

The point x is called a **limit of the net** α , we write $\lim_{\gamma \in \Gamma} \alpha(\gamma) = x$ or $\alpha(\gamma) \xrightarrow{\gamma \in \Gamma} x$.

Remark: If X is Hausdorff, then each net in X has at most one limit.

Proposition 15. *Let X be a topological space. Then:*

(1) *Let $A \subset X$. Then*

$$\bar{A} = \{x \in X; \exists \text{ a net } \alpha : \Gamma \rightarrow A : \alpha(\gamma) \xrightarrow{\gamma \in \Gamma} x\}$$

(2) *Let $A \subset X$. Then A is closed if and only if any $x \in X$, which is a limit of a net in A , belongs to A .*

(3) *Let Y be a topological space, $f : X \rightarrow Y$ a mapping and $x \in X$. The mapping f is continuous at x , if and only if*

$$\forall \text{ net } \alpha : \Gamma \rightarrow X : \alpha(\gamma) \xrightarrow{\gamma \in \Gamma} x \Rightarrow f(\alpha(\gamma)) \xrightarrow{\gamma \in \Gamma} f(x).$$