CONVOLUTIONS OF DISTRIBUTIONS

Recall : $U \in \mathcal{D}'(\mathbb{R}^d)$, $\psi \in \mathcal{D}(\mathbb{R}^d) \Rightarrow U * \psi(x) = U(\tau_x \check{\psi})$, $x \in \mathbb{R}^d$

Hence, the regular distribution induced by $U * \psi$ equals:

$$\Lambda_{U*\psi}(\varphi) = \int_{\mathbb{R}^d} U*\psi(x)\,\varphi(x)\,dx = \int_{\mathbb{R}^d} U(y \mapsto \psi(x-y))\,\varphi(x)\,dx =$$

$$= \int_{\mathbb{R}^d} U(y \mapsto \underbrace{\psi(x-y)\varphi(x)}_{\in \mathcal{D}(\mathbb{R}^d \times \mathbb{R}^d)})\,dx = \Lambda_1\big(x \mapsto U(y \mapsto \psi(x-y)\varphi(x))\big)$$

$\in \mathcal{D}(\mathbb{R}^d \times \mathbb{R}^d)$, clearly C^∞, $x \in \operatorname{spt}\varphi$, $x-y \in \operatorname{spt}\psi \Rightarrow y \in x - \operatorname{spt}\psi$
$\subset \operatorname{spt}\varphi - \operatorname{spt}\psi$

P VII.14 (b)

$$= U\big(y \mapsto \underbrace{\Lambda_1(x \mapsto \psi(x-y)\varphi(x))}_{= \int \psi(x-y)\varphi(x)\,dx = \int \check{\psi}(y-x)\varphi(x)\,dx = \check{\psi}*\varphi(y)}\big) = U(\check{\psi} * \varphi)$$

So, $\Lambda_{U*\psi}(\varphi) = U(\check{\psi} * \varphi)$.

Hence, it would be natural to define (for $U, V \in \mathcal{D}'(\Omega)$) :

$$U*V(\varphi) = U(\check{V} * \varphi), \quad \varphi \in \mathcal{D}(\Omega) \qquad (*)$$

But it is not possible in general, as $\check{V} * \varphi$ belongs to $C^\infty(\mathbb{R}^d)$, but not necessarily to $\mathcal{D}(\mathbb{R}^d)$.

Some cases when it is possible to define $U*V$ by formula $(*)$:

<u>Case 1:</u> $\operatorname{spt} V$ is compact.

Then, given $\varphi \in \mathcal{D}(\mathbb{R}^d)$, Theorem VII.15 (c) shows that $\operatorname{spt}(\check{V} * \varphi) \subset \operatorname{spt}\check{V} + \operatorname{spt}\varphi$, so it is compact. Therefore $\check{V} * \varphi \in \mathcal{D}(\mathbb{R}^d)$ and hence $(*)$ has a sense.

Moreover, $U * V \in \mathcal{D}'(\mathbb{R}^d)$ in this case :

Let $K \subset \mathbb{R}^d$ be compact. Let $L := K + \operatorname{spt} \check{V}$.
Then L is compact, hence $\exists N \in \mathbb{N}_0 \; \exists C > 0$:

$$|U(\psi)| \leq C \cdot \|\psi\|_N, \quad \psi \in \mathcal{D}_L(\mathbb{R}^d)$$

If $\varphi \in \mathcal{D}_K(\mathbb{R}^d)$, then $\check{V} * \varphi \in \mathcal{D}_L(\mathbb{R}^d)$

$\ulcorner$ $\operatorname{spt} \check{V} * \varphi \subset \operatorname{spt} \check{V} + \operatorname{spt} \varphi \subset \operatorname{spt} \check{V} + K = L$ $\lrcorner$

So,

$$|U * V(\varphi)| = |U(\check{V} * \varphi)| \leq C \cdot \|\check{V} * \varphi\|_N$$

Further, $\check{V} * \varphi(x) = \check{V}(y \mapsto \varphi(x-y)) = V(y \mapsto \varphi(x+y))$
Since V has compact support, by Prop. VII.12 (d) $\exists D > 0 \; \exists M \in \mathbb{N}_0$
s.t. $|V(\psi)| \leq D \|\psi\|_M$, $\psi \in \mathcal{D}(\mathbb{R}^d)$.

Fix α s.t. $|\alpha| \leq N$ and $x \in \mathbb{R}^d$. Then:

$$|D^\alpha(\check{V} * \varphi)(x)| \underset{\text{Thm VII.15 (b)}}{=} |(\check{V} * D^\alpha \varphi)(x)| = |V(y \mapsto D^\alpha \varphi(x+y))|$$

$$= |V(\tau_{-x} D^\alpha \varphi)| \leq D \cdot \|\tau_{-x} D^\alpha \varphi\|_M = D \cdot \|D^\alpha \varphi\|_M \leq D \cdot \|\varphi\|_{N+M}$$

So, $|(U * V)(\varphi)| \leq C \cdot D \cdot \|\varphi\|_{N+M}$ for $\varphi \in \mathcal{D}_K(\mathbb{R}^d)$.

Hence $U * V \in \mathcal{D}'(\mathbb{R}^d)$.

<u>Case 2</u>: $\operatorname{spt} U$ is compact

As in the proof of Prop VII.12 (d) we find $\psi \in \mathcal{D}(\mathbb{R}^d)$
s.t. $U(\varphi) = U(\psi \cdot \varphi)$ for $\varphi \in \mathcal{D}(\mathbb{R}^d)$

$\ulcorner$ $\psi \geq 0$ and $\psi \equiv 1$ on an open set containing $\operatorname{spt} V$ $\lrcorner$

Then one may naturally extend U to $C^\infty(\mathbb{R}^d)$ by setting

$$U(f) := U(\psi \cdot f), \quad f \in C^\infty(\mathbb{R}^d).$$

Then $(*)$ again has a sense.
Moreover, $U * V$ is a distribution:

U has cpt support $\overset{\text{Prop VII.12(d)}}{\Longrightarrow}$ $\exists\, C>0,\ N\in\mathbb{N}_0 : |U(\varphi)| \le C\cdot\|\varphi\|_N,\ \varphi\in\mathcal{D}(\mathbb{R}^d)$

So,

$|(U*V)(\varphi)| = |U(\psi\cdot(\check{V}*\varphi))| \le C\cdot\|\psi\cdot(\check{V}*\varphi)\|_N \quad ,\ \varphi\in\mathcal{D}(\mathbb{R}^d)$

Fix $K\subset\mathbb{R}^d$ compact. Let $\varphi\in\mathcal{D}_K(\mathbb{R}^d)$ and $x\in\operatorname{spt}\psi$.

Then $(\check{V}*\varphi)(x) = V(y\mapsto\varphi(x+y))$ [computed above]

$= V(\tau_{-x}\varphi)$

$\operatorname{spt}\tau_{-x}\varphi = \operatorname{spt}\varphi - x \subset K - \operatorname{spt}\psi =: L$

L is compact $\Rightarrow \exists\, D>0,\ M\in\mathbb{N}_0$ s.t. $|V(\eta)| \le D\|\eta\|_M,\ \eta\in\mathcal{D}_L(\mathbb{R}^d)$

So, $|\check{V}*\varphi(x)| \le D\|\tau_{-x}\varphi\|_M = D\|\varphi\|_M,\quad \varphi\in\mathcal{D}_K(\mathbb{R}^d),\ x\in\operatorname{spt}\psi$

α multiindex, $|\alpha|\le N \Rightarrow$

$|D^\alpha(\check{V}*\varphi)(x)| = |\check{V}*D^\alpha\varphi(x)| \le D\|D^\alpha\varphi\|_M \le D\|\varphi\|_{M+N},\ \varphi\in\mathcal{D}_K(\mathbb{R}^d),$
$x\in\operatorname{spt}\psi$

By the above we know that $\forall\,\varphi\in\mathcal{D}_K(\mathbb{R}^d)$:

$|U*V(\varphi)| \le C\|\psi\cdot(\check{V}*\varphi)\|_N \le C\cdot 2^N\cdot\|\psi\|_N\cdot\|\check{V}*\varphi\restriction_{\operatorname{spt}\psi}\|_N$

See the proof of Prop 8

$\le \underbrace{C\cdot 2^N\cdot\|\psi\|_N\cdot D}_{\text{a positive constant}}\cdot\|\varphi\|_{M+N}$

This shows $U*V\in\mathcal{D}'(\mathbb{R}^d)$.

<u>Case 3:</u> $\forall r>0$: $(\overline{U(0,r)} - \operatorname{spt}V)\cap\operatorname{spt}U$ is compact.

For $r>0$ fix $\psi_r\in\mathcal{D}(\mathbb{R}^d)$ s.t. $\psi_r\ge 0$ and $\psi_r = 1$ on an open set containing $(\overline{U(0,r)} - \operatorname{spt}V)\cap\operatorname{spt}U$.

We now extend U to the space

$Y = \{f\in C^\infty(\mathbb{R}^d);\ \exists\, r>0 : \operatorname{spt}f\subset\overline{U(0,r)} - \operatorname{spt}V\}$

by $\tilde{U}(f) = U(\psi_r f)$ if $\operatorname{spt} f \subset \overline{U(0,r)} - \operatorname{spt} V$

Clearly, Y is a vector space. Moreover, $\tilde{U}$ is a well-defined linear extension of U:

- $\psi_r f \in \mathcal{D}(\mathbb{R}^d)$, so $U(\psi_r f)$ is defined.
- Assume $f \in C^\infty(\mathbb{R}^d)$, $r > 0$, $\operatorname{spt} f \subset \overline{U(0,r)} - \operatorname{spt} V$, $s > r$.
  Then $(\psi_s - \psi_r) f \in \mathcal{D}(\mathbb{R}^d)$, we claim that $\operatorname{spt} (\psi_s - \psi_r) f \cap \operatorname{spt} U = \emptyset$

  $\ulcorner$ $G := \{x \in \mathbb{R}^d ; \psi_s = \psi_r = 1\}$. By the choice of ψ_s and ψ_r we know that $\operatorname{spt} U \cap (\overline{U(0,r)} - \operatorname{spt} V) \subset \operatorname{Int} G$

  Hence, $\operatorname{spt} (\psi_s - \psi_r) f \subset \operatorname{spt} (\psi_s - \psi_r) \cap \operatorname{spt} f \subset$
  $\subset (\mathbb{R}^d \setminus \operatorname{Int} G) \cap (\overline{U(0,r)} - \operatorname{spt} V)$

  Combining these two conclusions, we see that
  $$\operatorname{spt} (\psi_s - \psi_r) f \cap \operatorname{spt} U \subset (\mathbb{R}^d \setminus \operatorname{Int} G) \cap (\overline{U(0,r)} - G) \cap \operatorname{spt} U \subset$$
  $$\subset (\mathbb{R}^d \setminus \operatorname{Int} G) \cap \operatorname{Int} G = \emptyset. \lrcorner$$

  So, by Prop. VII.12 (c) we deduce $U((\psi_s - \psi_r) f) = 0$,
  i.e. $U(\psi_s f) = U(\psi_r f)$.

  Hence, $\tilde{U}$ is well defined.
- If $\varphi \in \mathcal{D}(\mathbb{R}^d)$, then $\operatorname{spt} \varphi \subset \overline{U(0,r)} - \operatorname{spt} V$ for r large enough,
  then $\psi_r \varphi = \varphi$, so $\tilde{U}(\varphi) = U(\psi_r \varphi) = U(\varphi)$.
  Thus, $\tilde{U}$ extends U.
- $\tilde{U}$ is linear ... clear: $f, g \in Y \Rightarrow \exists r > 0$ $\operatorname{spt} f, \operatorname{spt} g \subset \overline{U(0,r)} - \operatorname{spt} V$
  Then $\tilde{U}(f+g) = U(\psi_r(f+g)) = U(\psi_r f) + U(\psi_r g) = \tilde{U}(f) + \tilde{U}(g)$
  $\tilde{U}(\lambda f) = U(\psi_r \lambda f) = \lambda U(\psi_r f) = \lambda \tilde{U}(f)$

<u>Next:</u> $\varphi \in \mathcal{D}(\mathbb{R}^d) \Rightarrow \check{V} * \varphi \in Y$

$\ulcorner$ $\operatorname{spt} \check{V} * \varphi \subset \operatorname{spt} \check{V} + \operatorname{spt} \varphi = (-\operatorname{spt} V) + \operatorname{spt} \varphi \subset (-\operatorname{spt} V) + \overline{U(0,r)}$ for r large enough. $\lrcorner$

Thus $U * V$ may be defined by $U * V(\varphi) = \tilde{U}(\check{V} * \varphi)$

Clearly, it is a well-defined linear functional on $\mathcal{D}(\mathbb{R}^d)$

Moreover, it is a distribution on $\mathbb{R}^d$:

$K \subset \mathbb{R}^d$ compact, $\varphi \in \mathcal{D}_K(\mathbb{R}^d) \Rightarrow \operatorname{spt} \check{V} * \varphi \subset \operatorname{spt}\varphi - \operatorname{spt} V \subset K - \operatorname{spt} V$

Fix $r > 0$ s.t. $K \subset \overline{U(0,r)}$. then: $\varphi \in \mathcal{D}_K(\mathbb{R}^d) \Rightarrow \operatorname{spt} \check{V} * \varphi \subset \overline{U(0,r)} - \operatorname{spt} V$

Let $L := \operatorname{spt} \psi_r$. It is compact, so $\exists\, C, N$ s.t.

$$|U(\eta)| \le C \cdot \|\eta\|_N \text{ for } \eta \in \mathcal{D}_L(\mathbb{R}^d).$$

Then for $\varphi \in \mathcal{D}_K(\mathbb{R}^d)$ we have

$$|U * V(\varphi)| = |\check{U}(\check{V} * \varphi)| = |U(\psi_r \cdot (\check{V} * \varphi))| \le C \cdot \|\psi_r \cdot \check{V} * \varphi\|_N$$
$$\le C \cdot 2^N \cdot \|\psi_r\|_N \cdot \|\check{V} * \varphi_{\restriction L}\|_N$$

Next proceed similarly as in case 2:

$$x \in L \Rightarrow \check{V} * \varphi(x) = V(\tau_{-x}\varphi), \ \operatorname{spt} \tau_{-x}\varphi \subset K - L$$

Fix D, M s.t. $\forall\, \eta \in \mathcal{D}_{K-L}(\mathbb{R}^d): |V(\eta)| \le D\, \|\eta\|_M$

Then for $\varphi \in \mathcal{D}_K(\mathbb{R}^d)$ we have (as in Case 2):

$$|U * V(\varphi)| \le C \cdot 2^N \cdot \|\psi_r\|_N \, \|\check{V} * \varphi_{\restriction L}\|_N \le C \cdot 2^N \|\psi_r\|_N \, D \cdot \|\varphi\|_{M+N}$$

So, $U * V$ is a distribution.

<u>Case 4</u>: $\exists\, M, N \in \mathbb{N}_0$, $C, D > 0$ s.t. $|U(\varphi)| \le C\|\varphi\|_N$, $\varphi \in \mathcal{D}(\mathbb{R})$.
$|V(\varphi)| \le D\, \|\varphi\|_M$

Then $\exists\, \mu_\alpha$, $|\alpha| \le N$ finite Borel regular measures on $\mathbb{R}^d$ (signed or complex)

s.t. $U(\varphi) = \sum_{|\alpha| \le N} \int_{\mathbb{R}^d} D^\alpha \varphi \, d\mu_\alpha$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$

[Consider $T : \mathcal{D}(\mathbb{R}^d) \to C_0(\mathbb{R}^d)^{\{\alpha,\, |\alpha| \le N\}}$ defined by $T\varphi = (D^\alpha \varphi)_{|\alpha| \le N}$

Then T is a linear bijection of $\mathcal{D}(\mathbb{R}^d)$ onto $Z := T(\mathcal{D}(\mathbb{R}^d))$

Moreover, $U \circ T^{-1}$ is a continuous linear functional on Z equipped with the norm inherited from $C_0(\mathbb{R}^d)^{\{\alpha,\, |\alpha| \le N\}}$ ($\|(f_\alpha)_\alpha\| = \max_\alpha \|f_\alpha\|_\infty$), $\|U \circ T^{-1}\| \le C$

By Hahn–Banach thm we extend $U\circ T^{-1}$ to a cts linear functional $L : (C_0(\mathbb{R}))^{\{\alpha \mid |\alpha| \leq N\}} \to \mathbb{F}$. By Riesz representation thm $\exists\, \mu_\alpha, |\alpha| \leq N$ finite Borel $\mathbb{F}$-valued measures on $\mathbb{R}$ s.t. $L((f_\alpha)_\alpha) = \sum_\alpha \int_{\mathbb{R}^d} f_\alpha \, d\mu_\alpha$

Then, given $\varphi \in \mathcal{D}(\mathbb{R}^d)$,
$$U(\varphi) = U\circ T^{-1}(T\varphi) = L\left((D^\alpha \varphi)_{|\alpha| \leq N}\right) = \sum_{|\alpha| \leq N} \int_{\mathbb{R}^d} D^\alpha \varphi \, d\mu_\alpha$$

Then U may be extended to $Y := \{ f \in C^\infty(\mathbb{R}^d) ; \|f\|_N < \infty \}$ by setting
$$\tilde{U}(f) = \sum_{|\alpha| \leq N} \int_{\mathbb{R}^d} D^\alpha f \, d\mu_\alpha \quad , \quad f \in Y$$

<u>Remark</u>: The choice of $(\mu_\alpha)_{|\alpha| \leq N}$ need not be unique (H-B thm is used and, moreover, there are examples – for $d=1$, $N=1$ the pairs $(\delta_1 - \delta_0, 0)$ and $(0, \lambda|_{[0,1]})$, where λ is the Lebesgue measure represent the same distribution).

However, $\tilde{U}$ is uniquely determined by U:

Assume $\tilde{U}$ in the above form and $U = \tilde{U}|_{\mathcal{D}(\mathbb{R}^d)} = 0$. Let $\varepsilon > 0$ be arbitrary.

Find $R > 1$ s.t. $|\mu_\alpha|(\mathbb{R}^d \setminus U(0,R)) < \varepsilon$ for each $|\alpha| \leq N$

Fix $\psi \in \mathcal{D}(\mathbb{R}^d)$, $\psi \geq 0$, $\psi = 1$ on $\overline{U(0,1)}$

Let $\psi_R(x) = \psi\left(\frac{x}{R}\right)$, $x \in \mathbb{R}^d \Rightarrow \psi_R \in \mathcal{D}(\mathbb{R}^d)$, $\psi_R = 1$ on $\overline{U(0,R)}$

$D^\alpha \psi_R(x) = \frac{1}{R^{|\alpha|}} D^\alpha \psi\left(\frac{x}{R}\right) \Rightarrow \|D^\alpha \psi_R\|_\infty \leq \|D^\alpha \psi\|$ (recall $R > 1$)

Let $f \in Y$ be arbitrary. Then $\psi_R f \in \mathcal{D}(\Omega) \Rightarrow \tilde{U}(\psi_R f) = 0$

$\Rightarrow$
$$|\tilde{U}(f)| = |\tilde{U}((1-\psi_R)f)| = \left| \sum_{|\alpha| \leq N} \int_{\mathbb{R}^d} D^\alpha((1-\psi_R)f) \, d\mu_\alpha \right|$$

$$\leq \sum_{|\alpha| \leq N} \int_{\mathbb{R}^d \setminus U(0,R)} |D^\alpha (1-\psi_R) f| \, d|\mu_\alpha| = (*)$$

$$|D^\alpha (1-\psi_R) f| \leq \sum_{\beta \leq \alpha} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d} |D^\beta (1-\psi_R) D^{\alpha-\beta} f|$$

$$\leq \sum_{\beta \leq \alpha} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d} \|1-\psi\|_N \|f\|_N = 2^N \|1-\psi\|_N \|f\|_N$$

So, $(*) \leq \sum_{|\alpha| \leq N} 2^N \|1-\psi_R\|_N \|f\|_N \cdot \varepsilon$

. Since $\varepsilon > 0$ is arbitrary, $\tilde{U}(f) = 0$. $\lrcorner$

Moreover, for $\varphi \in \mathcal{D}(\mathbb{R}^d)$ we have (for $x \in \mathbb{R}^d$)

$$|\check{V} * \varphi(x)| = |V(\tau_{-x}\varphi)| \leq D \|\tau_{-x}\varphi\|_M = D\|\varphi\|_M$$

and if α is a multiindex, then

$$|D^\alpha(\check{V} * \varphi)(x)| = |\check{V} * D^\alpha\varphi(x)| \leq D \cdot \|D^\alpha \varphi\|_M \leq D\|\varphi\|_{|\alpha|+M}$$

Hence, $\|\check{V} * \varphi\|_N \leq D\|\varphi\|_{N+M}$ $(< \infty)$, so $\check{V} * \varphi \in \mathcal{Y}$

$\Rightarrow U * V(\varphi) = \tilde{U}(\check{V} * \varphi)$ may be defined.

Clearly, it is a linear functional on $\mathcal{D}(\mathbb{R}^d)$. Moreover,

$$|U * V(\varphi)| = |\tilde{U}(\check{V} * \varphi)| \leq C \cdot \|\check{V} * \varphi\|_N \leq C \cdot D \cdot \|\varphi\|_{N+M}$$

So, $U * V$ is a distribution.

Prop VII.16 (a) In Cases 1–4 we get a distribution. Moreover, Case 3 covers Cases 1 and 2; further $U * V = V * U$

$\ulcorner$ Above we have proved that $U * V$ is a distribution.
It is clear that, in case $\operatorname{spt} U$ or $\operatorname{spt} V$ is compact, Case 3 may be used and that the result is the same as in Case 1 or 2.
Further, if Case 3 is applicable to U, V, it is applicable to V, U as well and $V * U = U * V$:

$$\left(\overline{U(0,R)} - \operatorname{spt} U\right) \cap \operatorname{spt} V = \overline{U(0,R)} + \left(-\operatorname{spt} U\right) \cap \left(\operatorname{spt} V - \overline{U(0,R)}\right)$$

$$= \overline{U(0,R)} - \operatorname{spt} U \cap \left(\overline{U(0,R)} - \operatorname{spt} V\right)$$

So, if the set is compact, also is compact.

Let ψ_R be as in Case 3 – it may be chosen common for pairs U, V and V, U

$$U * V(\varphi) = \tilde{U}(\check{V} * \varphi) \underset{\substack{\uparrow \\ \operatorname{spt} \varphi \subset U(0,R)}}{=} U(\psi_R \cdot \check{V} * \varphi) = U\left(x \mapsto \psi_R(x) V(y \mapsto \varphi(x+y))\right) =$$

$$= U\left(x \mapsto V\left(y \mapsto \psi_R(x)\varphi(x+y)\right)\right) \overset{\text{Prop VIII.14 (b)}}{=} V\left(y \mapsto U\left(x \mapsto \psi_R(x)\varphi(x+y)\right)\right) = (*)$$

Let us analyze the last expression:

$$U\left(x \mapsto \psi_R(x)\varphi(x+y)\right) \neq 0 \Rightarrow \operatorname{spt}(\psi_R \cdot \tau_{-y}\varphi) \cap \operatorname{spt} U \neq \emptyset$$
$$\Rightarrow \operatorname{spt}(\tau_{-y}\varphi) \cap \operatorname{spt} U \neq \emptyset \Rightarrow (\operatorname{spt}\varphi - y) \cap \operatorname{spt} U \neq \emptyset$$
$$\Rightarrow y \in \operatorname{spt}\varphi - \operatorname{spt} U \subset U(0,R) - \operatorname{spt} U$$
$$\Rightarrow \operatorname{spt}\left(y \mapsto U(x \mapsto \psi_R(x)\varphi(x+y))\right) \subset \overline{U(0,R)} - \operatorname{spt} U$$
$$\Rightarrow (*) = V\left(y \mapsto \psi_R(y)\, U(x \mapsto \psi_R(x)\varphi(x+y))\right) =$$
$$= V\left(y \mapsto U(x \mapsto \psi_R(x)\psi_R(y)\varphi(x+y))\right)$$

Similarly, by exchanging roles of U, V, we prove
$V * U(\varphi) = U\left(x \mapsto V(y \mapsto \psi_R(y)\psi_R(x)\varphi(x+y))\right)$ if $\operatorname{spt}\varphi \subset U(0,R)$

The two formulas coincide by Prop. VIII.14 (b).

- If U, V satisfy Case 4, then $U * V = V * U$

Both U and V may be represented using finite families of measures and canonically extended to the respective spaces Y_U, Y_V

Then $U * V(\varphi) = \tilde{U}(\check{V} * \varphi) = \tilde{U}\left(x \mapsto V(y \mapsto \varphi(x+y))\right) =$

$$= \tilde{U}\left(x \mapsto \tilde{V}(y \mapsto \varphi(x+y))\right) = \tilde{V}\left(y \mapsto \tilde{U}(x \mapsto \varphi(x+y))\right) =$$

$(x,y) \mapsto \varphi(x+y)$ is C^∞ with bdd derivatives
use Fubini thm as in the proof of P VIII.14 (b)

$= \dots = V * U(\varphi)$.

(b) $V = \Lambda_\varphi$ for some $\varphi \in \mathcal{D}(\mathbb{R}^d) \Rightarrow U * V = \Lambda_{U * \varphi}$

See the computation motivating formula $(*)$

(c) $U = \Lambda_f$, $f \in L^1_{loc}(\mathbb{R}^d)$, $V = \Lambda_\varphi$, $\varphi \in \mathcal{D}(\mathbb{R}^d) \Rightarrow U * V = \Lambda_{f * \varphi}$

$U * V \overset{(b)}{=} \Lambda_{U * \varphi} = \Lambda_{f * \varphi}$

Thm VIII.15 (a)

(d) $U = \Lambda_f,\ V = \Lambda_g,\ f,g \in \mathcal{L}^1(\mathbb{R}^d) \Rightarrow U * V = \Lambda_{f*g}$

⌈ Case 4 applies:

Thm VI.15 (a)

$$U * V(\varphi) = \tilde{U}(\check{V} * \varphi) = \tilde{U}(\Lambda_{\check g} * \varphi) = \tilde{U}(\check{g} * \varphi) =$$

Case 4 formula ($d\mu = f\,dx$, $\nu = 0$)

$$= \int_{\mathbb{R}^d} f(x)\cdot(\check{g}*\varphi)(x)\,dx = \int_{\mathbb{R}^d} f(x)\Big(\int_{\mathbb{R}^d} \underbrace{\check{g}(x-y)}_{=g(y-x)}\varphi(y)\,dy\Big)dx =$$

$$\overset{\text{FUBINI THM}}{=} \int_{\mathbb{R}^d} \varphi(y)\int_{\mathbb{R}^d} f(x)g(y-x)\,dx\,dy = \int_{\mathbb{R}^d} \varphi(y)(f*g)(y)\,dy = \Lambda_{f*g}(\varphi)$$

⌈ FUBINI THM

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\varphi(y)f(x)g(y-x)|\,dx\,dy \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|\varphi\|_\infty |f(x)|\,|g(y-x)|\,dy\,dx \le \|\varphi\|_\infty \|f\|_{L^1}\|g\|_{L^1}$$ ⌋ ⌋

(e) $\operatorname{spt}(U * V) \subset \operatorname{spt} U + \operatorname{spt} V$

⌈ Case 3: Assume that $\operatorname{spt}\varphi \cap (\operatorname{spt} U + \operatorname{spt} V) = \emptyset$
and $\operatorname{spt}\varphi \subset \overline{U(0,R)}$

$$U * V(\varphi) = \tilde{U}(\check{V} * \varphi) = U(\varphi_R \cdot (\check{V} * \varphi))$$

$\check{V} * \varphi(x) = V(y \mapsto \varphi(x+y))$ ... $\operatorname{spt}\check{V} * \varphi \subset \operatorname{spt}\varphi - \operatorname{spt} V$

$\Rightarrow \operatorname{spt}(\varphi_R\cdot(\check{V}*\varphi)) \subset \operatorname{spt}\varphi - \operatorname{spt} V$

Moreover, $(\operatorname{spt}\varphi - \operatorname{spt} V) \cap \operatorname{spt} U = \emptyset$

so $U(\varphi_R(\check{V}*\varphi)) = 0$ by Prop VII.12 (c)

Hence $U * V(\varphi) = 0$.

We deduce that $U * V$ vanishes on $\mathbb{R}^d \setminus (\operatorname{spt} U + \operatorname{spt} V)$

<u>Case 4</u>: Assume that $\operatorname{spt}\varphi \cap (\operatorname{spt} U + \operatorname{spt} V) = \emptyset$. As above, $\operatorname{spt}\check{V}*\varphi \subset \operatorname{spt}\varphi - \operatorname{spt} V$
and $(\operatorname{spt}\varphi - \operatorname{spt} V) \cap \operatorname{spt} U = \emptyset$.

Let $\psi \in \mathcal{D}(\mathbb{R}^d)$, $\psi = 1$ on $\overline{U(0,1)}$, $\psi \ge 0$

For $R > 0$ let $\psi_R(x) = \psi(\frac{x}{R})$. Then

$U(\psi_R(\check{V}*\varphi)) = 0$ for each $R > 0$

Moreover, the proof of Case 4 yields $\tilde{U}(\check{V} * \varphi) = \lim U(\psi_R(\check{V} * \varphi)) = 0$ ┘

(f) $D^\alpha(U * V) = D^\alpha U * V = U * D^\alpha V$

<u>Case 3</u>: $D^\alpha(U * V)(\varphi) = (-1)^{|\alpha|}(U * V)(D^\alpha \varphi) =$

$= (-1)^{|\alpha|} U(\psi_R \cdot (\check{V} * D^\alpha \varphi)) \underset{\text{Thm VII.15 (3)}}{=} (-1)^{|\alpha|} U(\psi_R \cdot (D^\alpha \check{V} * \varphi)) =$

$= U(\psi_R \cdot (D^\alpha V)^\vee * \varphi) = (U * D^\alpha V)(\varphi)$

$D^\alpha \check{V}(\eta) = \check{V}(D^\alpha \eta) = V(x \mapsto D^\alpha \eta(-x)) = V((-1)^{|\alpha|} D^\alpha \check{\eta})$

Using (a): $D^\alpha(U * V) = D^\alpha(V * U) = V * D^\alpha U = D^\alpha U * V$

<u>Case 4</u>: $D^\alpha(U * V)(\varphi) = (-1)^{|\alpha|} U * V(D^\alpha \varphi) = (-1)^{|\alpha|} \tilde{U}(\check{V} * D^\alpha \varphi) =$

$= (-1)^{|\alpha|} \tilde{U}(D^\alpha \check{V} * \varphi) = \tilde{U}((D^\alpha V)^\vee * \varphi) = U * D^\alpha V(\varphi)$

the last formula as above.

(g) $U = U * 1_{\delta_0}$, $D^\alpha U = U * D 1_{\delta_0}$

⌈ $U * 1_{\delta_0}(\varphi) = U(\check{1}_{\delta_0} * \varphi) = U(1_{\delta_0} * \varphi) = U(\varphi)$

⌈ $1_{\delta_0} * \varphi(x) = 1_{\delta_0}(y \mapsto \varphi(x-y)) = \varphi(x-0) = \varphi(x)$ ┘┘

$D^\alpha U = D^\alpha(U * 1_{\delta_0}) \overset{(f)}{=} U * D^\alpha 1_{\delta_0}$ ┘

(Case 1 applies)