

VI. Weak topologies

VI.1 General weak topologies and duality

Definition. Let X be a vector space over \mathbb{F} .

- By $X^\#$ we denote the **algebraic dual** of X , i.e., the vector space of all the linear functionals $f : X \rightarrow \mathbb{F}$.
- Let $M \subset X^\#$ be a nonempty set. By $\sigma(X, M)$ we denote the topology on X generated by the family of seminorms
$$\{x \mapsto |f(x)|; f \in M\}.$$

It is called **the weak topology generated by M** .

Proposition 1.

- (1) *The space X is a LCS if it is equipped by the topology $\sigma(X, M)$.*
- (2) *The topology $\sigma(X, M)$ is Hausdorff if and only if M **separates points of X** , i.e., if and only if for any $x \in X \setminus \{\mathbf{o}\}$ there exists $f \in M$ satisfying $f(x) \neq 0$.*
- (3) *Functionals from M are continuous on $(X, \sigma(X, M))$.*
- (4) *$\sigma(X, M)$ is the weakest (i.e., the smallest) topology on X in which all the functionals from M are continuous.*
- (5) *$\sigma(X, M) = \sigma(X, \text{span } M)$.*
- (6) *Let T be a topological space and let $F : T \rightarrow X$ be any mapping. Then F is a continuous mapping of T to $(X, \sigma(X, M))$ if and only if $f \circ F$ is continuous on X for each $f \in M$.*

Examples 2.

- (1) *Let X be a LCS. Then $X^* \subset\subset X^\#$, the topology $\sigma(X, X^*)$ is called **the weak topology of X** , sometimes it is denoted by w . If X is Hausdorff, the topology $\sigma(X, X^*)$ is Hausdorff as well.*
- (2) *Let X be a LCS. Define a mapping $\varkappa : X \rightarrow (X^*)^\#$ by
$$\varkappa(x)(f) = f(x), f \in X^*, x \in X.$$
Then $\varkappa(X)$ is a subspace of $(X^*)^\#$ separating points of X^* , hence the topology $\sigma(X^*, \varkappa(X))$ is Hausdorff. It is called **the weak* topology of X^*** , it is denoted by $\sigma(X^*, X)$ or by w^* .*
- (3) *Let Γ be a nonempty set and let the space \mathbb{F}^Γ be equipped by the product topology (cf. Example V.1(2)). The product topology equals $\sigma(\mathbb{F}^\Gamma, M)$ where $M = \{\mathbf{x} \mapsto \mathbf{x}(\gamma); \gamma \in \Gamma\}$.*

- (4) Let T be a topological space and let $\mathcal{C}(T, \mathbb{F})$ be the vector space of all the continuous functions on T . For $t \in T$ define the functional $\varepsilon_t \in \mathcal{C}(T, \mathbb{F})^\#$ by the formula

$$\varepsilon_t(f) = f(t), \quad f \in \mathcal{C}(T, \mathbb{F}).$$

Then $M = \{\varepsilon_t; t \in T\}$ is a subset of $\mathcal{C}(T, \mathbb{F})^\#$ separating points of $\mathcal{C}(T, \mathbb{F})$, the topology $\sigma(\mathcal{C}(T, \mathbb{F}), M)$ is therefore Hausdorff. It is called **the topology of pointwise convergence**, it is denoted by τ_p or by $\tau_p(T)$.

- (5) Using the notation from the previous item, let moreover $D \subset T$ be a nonempty set and $M_D = \{\varepsilon_t; t \in D\}$. The topology $\sigma(\mathcal{C}(T, \mathbb{F}), M_D)$ is called **the topology of pointwise convergence on D** , it is denoted by $\tau_p(D)$. If D is dense in T , then the topology $\tau_p(D)$ is Hausdorff.

Lemma 3. Let X be a vector space and $f, f_1, \dots, f_k \in X^\#$. The following assertions are equivalent:

- (i) $f \in \text{span}\{f_1, \dots, f_k\}$;
- (ii) $\exists C > 0 \forall x \in X : |f(x)| \leq C \cdot \max\{|f_1(x)|, \dots, |f_k(x)|\}$;
- (iii) $\bigcap_{j=1}^k \text{Ker } f_j \subset \text{Ker } f$.

Theorem 4. Let X be a vector space and let $M \subset X^\#$ be a nonempty set. Then $(X, \sigma(X, M))^* = \text{span } M$.

Corollary 5.

- (a) Let X be a LCS and let $f \in X^\#$. Then f is continuous on X (i.e., $f \in X^*$), if and only if it is weakly continuous (i.e., $\sigma(X, X^*)$ -continuous) on X .
- (b) Let X be a LCS. Then $(X^*, \sigma(X^*, X))^* = \varkappa(X)$ (cf. Example 2(2)).
- (c) Let X be a normed linear space and let $f \in X^{**}$. Then $f \in \varkappa(X)$ (where $\varkappa : X \rightarrow X^{**}$ is the canonical embedding), if and only if f is weak* continuous (i.e., $\sigma(X^*, X)$ continuous) on X^* .

Remark: Weak topologies are, by definition, locally convex. However, some results from this section may be applied for TVS as well:

- If X is a TVS, the weak topology on X may be defined to be $\sigma(X, X^*)$, in the same way as in the locally convex case. The weak topology is locally convex, but it need not be Hausdorff even if X is Hausdorff.
- If X is a TVS, one may define the mapping \varkappa as above and define the weak* topology on X^* as $\sigma(X^*, \varkappa(X))$. But this is not very useful, because the dual may be very small (even trivial, see the final remarks in Section V.6) and \varkappa need not be one-to-one even if X is Hausdorff.
- Assertions (a) and (b) from Corollary 5 hold as well (no change needed).