Theorem X.46 Let H be a real Hilbert space

$$H_C = H + iH = \{x + iy \,;\, x, y \in H\}$$

$$\langle x+iy, u+iv\rangle := \langle x,u\rangle + \langle y,v\rangle + i\left(\langle y,u\rangle - \langle x,v\rangle\right)$$
$x+iy,\ u+iv \in H_C$

It is an inner product and H_C is a complex Hilbert space

Let T be an operator on H (with domain $D(T)$)
Define $T_C(x+iy) := T(x) + iT(y)$, $x+iy \in D(T_C) = \{x+iy,\ x,y \in D(T)\}$

Then: (1) T_C is a linear operator, $D(T_C)$ is a linear subspace
(2) $D(T)$ dense $\Rightarrow$ $D(T_C)$ dense in H_C
(3) Suppose T densely defined $\Rightarrow$ $(T_C)^* = (T^*)_C$

$\supseteq$: $u,v \in D(T^*)$, $x+iy \in D(T_C)$

$$\langle T_C(x+iy), u+iv\rangle = \langle Tx + iTy, u+iv\rangle =$$
$$= \langle Tx,u\rangle + \langle Ty,v\rangle + i\left(\langle Ty,u\rangle - \langle Tx,v\rangle\right) =$$
$$= \langle x,T^*u\rangle + \langle y,T^*v\rangle + i\left(\langle y,T^*u\rangle - \langle x,T^*v\rangle\right) =$$
$$= \langle x+iy, T^*u + iT^*v\rangle = \langle x+iy, (T^*)_C(u+iv)\rangle$$

$\subseteq$: $u+iv \in D((T_C)^*)$

$\Rightarrow \exists\ w+iz \in H_C$ s.t. $\forall x+iy \in D(T_C)$:

$$\langle T_C(x+iy), u+iv\rangle = \langle x+iy, w+iz\rangle$$

So, $\langle Tx,u\rangle + \langle Ty,v\rangle = \langle x,w\rangle + \langle y,z\rangle$, $x,y \in D(T)$
(take the real part). apply for $y=0$ $\Rightarrow u \in D(T^*)$, $T^*u = w$
apply for $x=0$ $\Rightarrow v \in D(T^*)$, $T^*v = z$.

(4) T self-adjoint $\Rightarrow$ T_C self-adjoint

(clear from (3))

(5) T self-adjoint, $\lambda \in \mathbb{R}$ $\Rightarrow$ ($\lambda I - T$ invertible $\Leftrightarrow$ $(\lambda I - T_C)$ invertible)

- $(\lambda I - T)x = 0 \Rightarrow (\lambda I - T_C)(x + i0) = 0$
- $(\lambda I - T_C)(x + iy) = 0$

$\Rightarrow (\lambda I - T)x + i(\lambda I - T)y = 0 \Rightarrow (\lambda I - T)x = 0$
$\& \ (\lambda I - T)y = 0$

So, $\lambda I - T$ is one-to-one $\Leftrightarrow$ $(\lambda I - T_C)$ is one-to-one

- $(\lambda I - T_C)(x + iy) = (\lambda I - T)x + i(\lambda I - T)y$

So, $(\lambda I - T_C)$ is onto $\Leftrightarrow$ $\lambda I - T$ is onto

(6) Suppose T is bdd and selfadjoint. Then $\forall f \in \mathcal{C}(\sigma(T_c))$ real-valued : $\tilde{f}(T_c)H \subset H$

[It holds for $f(t) = t^n, n \in \mathbb{N} \cup \{0\}$, by Stone-Weierstrass then it holds for $f \in \mathcal{C}(\sigma(T_c))$]

(7) The same holds for f bdd $\mathcal{B}_{T_c}$-measurable, real-valued

[Suppose $\tilde{f}(T_c)(H) \not\subset H$, so $\exists x \in H$

s.t. $\tilde{f}(T_c)(x) = u + iv,\quad u, v \in H,\ v \neq 0$

$\Rightarrow \langle \tilde{f}(T_c)x, v\rangle = \langle u, v\rangle + i\langle v, v\rangle \in \mathbb{C}\setminus\mathbb{R}$

$\exists (g_n)$ cts, $g_n \to f$ $|E_{x,v}|$-a.e., $\|g_n\|_\infty \leq \|f\|_\infty$

Then $\underbrace{\langle \tilde{g}_n(T_c)x, v\rangle}_{\in \mathbb{R}} \to \langle \tilde{f}(T_c)x, v\rangle \in \mathbb{C}\setminus\mathbb{R}$

Lebesgue dom. conv. th

a contradiction]

(8) So, if T is bdd, self-adjoint, ~~then~~ and E is the spectral measure of T_c, then $E(A)H \subset H$ for $A \in \mathcal{A}^{\mathbb{C}}_{\sigma}$. Therefore $E_R(A) := E(A)|_H$ defines a „real spectral measure"

and $T = \int \mathrm{id}\, dE_R$, since $\langle Tx, y\rangle = \int \mathrm{id}\, dE_{x,y} = \int \mathrm{id}\, d(E_R)_{x,y}$

$(E_R)_{x,y} = E_{x,y}$ for $x, y \in H$

(9) T unbdd self-adjoint $\Rightarrow T_C^2$ is selfadjoint, $T_C^2(H) \subset H$

$I + T_C^2$ also selfadjoint, maps H into H, $D(T_C^2)$ onto H_C

So, it maps $D(T_C^2) \cap H$ onto H

(see L 38)

$B := (I + T_C^2)^{-1} \in \mathcal{L}(H_C)$, it is selfadjoint and $B(H) \subset H$

$C := T_C B = T_C (I + T_C^2)^{-1} \in \mathcal{L}(H_C)$, selfadjoint, $C(H) \subset H$

$P_j := \widetilde{\chi}_{(\frac{1}{j+1}, \frac{1}{j}]}(B) \Rightarrow P_j(H) \subset H$ by (8)

Then TP_j is a bdd self-adjoint operator (see the proof of Th 50)

So, E_j, the spectral measure of TP_j satisfies $E_j(A) H \subset H$, $A \in \mathcal{A}_j$

Since $E = \sum E_j$ is the spectral measure of T, we see

$E(A) H \subset H$, $A \in \mathcal{A}$

(10) HENCE: $E_R(A) := E(A)\restriction_H$ defines a „real spectral measure" in H and $T = \int \mathrm{id}\, dE_R$,

Since $x_{ij} \in D(T) \Rightarrow$

$$\langle (\int \mathrm{id}\, dE_R) x_{ij} \rangle = \int \mathrm{id}\, d(E_R)_{x_{ij}} = \int \mathrm{id}\, dE_{x_{ij}} =$$

$$= \langle T_C x_{ij} \rangle = \langle T x_{ij} \rangle$$

($(E_R)_{x_{ij}} = E_{x_{ij}}$, $x_{ij} \in H$)