III.4 Hartogs theorem on separate holomorphy

Theorem 15 (Hartogs theorem). *Any separately holomorphic function is holomorphic.*

Remark: Locally bounded separately holomorphic functions are holomorphic by Theorem 8. A separately continuous function on $\mathbb{C}^2$ need not be locally bounded.

Lemma 16. *Let $G \subset \mathbb{C}$ be and open set and u be a subharmonic function on G. If $a \in G$ and $r > 0$ are such that $\overline{U(a,r)} \subset G$, then*

$$u(a) \leq \frac{1}{\pi r^2} \int_{U(a,r)} u \, \mathrm{d}\lambda,$$

where λ denotes the two-dimensional Lebesgue measure in $\mathbb{C}$.

Lemma 17. *Let $G \subset \mathbb{C}$ be and open set and (u_k) be a sequence of subharmonic functions on G, which is uniformly bounded from above on G. Let $C \in \mathbb{R}$ be such that*

$$\limsup_{k \to \infty} u_k(z) \leq C \text{ for each } z \in G.$$

Then for any compact set $K \subset G$ and any $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\forall z \in K \; \forall k \geq k_0 : u_k(z) \leq C + \varepsilon.$$

Lemma 18. *Let $n \geq 2$ be such that the statement of the Hartogs theorem holds for $n-1$. Let $\boldsymbol{a} = (\boldsymbol{a}', a_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$, $r, s, \varepsilon \in (0, \infty)$ and $\varepsilon < r$. Let f be a separately holomorphic function on the polydisc $\mathbb{P}(\boldsymbol{a}', \widehat{\boldsymbol{r}}) \times U(a_n, s)$, which is bounded on $\mathbb{P}(\boldsymbol{a}', \widehat{\boldsymbol{\varepsilon}}) \times U(a_n, s)$. Then f is holomorphic on $\mathbb{P}(\boldsymbol{a}', \widehat{\boldsymbol{r}}) \times U(a_n, s)$.*

(We use the notation $\widehat{\boldsymbol{r}} = (r, \ldots, r) \in (0, \infty)^{n-1}$.)