I.3 Hardy spaces on the unit disc

Definition. Let $\Omega \subset \mathbb{C}$ be an open set and $u : \Omega \to [-\infty, +\infty)$ be a function. The function u is said to be **subharmonic**, if it is upper semicontinuous and, moreover, whenever $a \in \Omega$ and $R > 0$ are such that $\overline{U(a,R)} \subset \Omega$, it holds

$$u(a) \le \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a + Re^{it}) \, \mathrm{d}t$$

and the integral on the right-hand side is not equal to $-\infty$.

Remark: Similarly one can define **superharmonic** functions (they are lower semicontinuous, have values in $(-\infty, +\infty]$, satisfy the opposite inequality and the respective integrals are not $+\infty$). Then a function is harmonic if and only if it is simultaneously subharmonic and superharmonic.

Theorem 13. *Let $\Omega \subset \mathbb{C}$ be a domain and let f be a holomorphic function on Ω which is not the constant zero function. Then the functions $\log|f|$, $\log^+|f|$ a $|f|^p$ ($p \in (0, +\infty)$) are subharmonic on Ω.*

Remark: In the above theorem we set $\log 0 = -\infty$ and $\log^+ t = (\log t)^+ = \max\{\log t, 0\}$ for $t \in [0, \infty)$.

Theorem 14. *Let $\Omega \subset \mathbb{C}$ be an open set and u be a subharmonic function on Ω. Let $a \in \Omega$ and $R > 0$ be such that $\overline{U(a,R)} \subset \Omega$. Let h be a function continuous on $\overline{U(a,R)}$ and harmonic on $U(a,R)$. If $u \le h$ on the circle $|z - a| = R$, then $u \le h$ on $U(a,R)$.*

Notation:

- $\mathbb{D} = U(0,1) = \{z \in \mathbb{C} : |z| < 1\}$

For $f \in H(\mathbb{D})$ and $r \in [0,1)$ set:

- $M_0(f,r) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ \left|f(re^{i\theta})\right| \, \mathrm{d}\theta\right)$
- $M_p(f,r) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(re^{i\theta})\right|^p \, \mathrm{d}\theta\right)^{1/p} \quad (0 < p < \infty)$
- $M_\infty(f,r) = \sup_{\theta \in [-\pi,\pi)} \left|f(re^{i\theta})\right|$

Theorem 15. *Let $f \in H(\mathbb{D})$.*

- *The function $r \mapsto M_p(f,r)$ is non-decreasing on $[0,1)$ for any $p \in [0,\infty]$.*
- *The function $p \mapsto M_p(f,r)$ is non-decreasing on $(0,\infty]$ for any $r \in (0,1)$.*
- *$M_0(f,r)^p \le 1 + M_p(f,r)^p$ for any $p \in (0,\infty)$ and $r \in (0,1)$.*

Definition.

- For $f \in H(\mathbb{D})$ and $p \in [0,\infty]$ set

$$\|f\|_p = \sup_{r \in [0,1)} M_p(f,r) = \lim_{r \to 1-} M_p(f,r).$$

- For $p \in (0,\infty]$ set

$$H^p = \{f \in H(\mathbb{D}) : \|f\|_p < \infty\}.$$

- Furher, set

$$N = \{f \in H(\mathbb{D}) : \|f\|_0 < \infty\}.$$

Remark. $H^p \subset H^s \subset N$ whenever $0 < s < p \le \infty$.

Lemma 16. *Let $f \in N$. Then there are $g, h \in H(\mathbb{D})$ such that $\|g\|_\infty \le 1$, h has no roots in $\mathbb{D}$, $h \in N$ and $\|h\|_p = \|f\|_p$ for each $p \in [0,\infty]$.*

Lemma 17. *Let $f \in H^p$.*

- *If $p \ge 1$, then $M_\infty(f,r) \le \frac{1}{1-r}\|f\|_1 \le \frac{1}{1-r}\|f\|_p$ for $r \in (0,1)$.*
- *If $p \in (0,1)$, then $M_\infty(f,r) \le \frac{3}{(1-r)^{1+\frac{1}{p}}}\|f\|_p$ for $r \in (0,1)$.*

Theorem 18.

- $(H^p, \|\cdot\|_p)$ *is a Banach space for any* $p \in [1, \infty]$.
- *If* $p \in (0,1)$, *then* H^p *is a complete metric linear space with the metric defined by the formula* $\rho_p(f,g) = \|f - g\|_p^p$.

Theorem 19. *Let* $f \in H(\mathbb{D})$ *satisfy*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad z \in \mathbb{D}.$$

Then

$$\|f\|_2^2 = \sum_{n=0}^{\infty} |a_n|^2.$$

If moreover $f \in H^2$, *then the following assertions hold:*

(1) *The limit* $f^*(e^{it}) = \lim\limits_{r\to 1-} f(re^{it})$ *exists for almost all* $t \in [0, 2\pi)$.
(2) $f^* \in L^2(\mathbb{T})$
(3) *For* $n \in \mathbb{Z}$ *define* $\varphi_n(e^{it}) = e^{int}$, $t \in [-\pi, \pi)$. *Then* $(\varphi_n)_{n\in\mathbb{Z}}$ *is an orthonormal basis of* $L^2(\mathbb{T})$ *and the expansion of* f^* *with respect to this basis is*

$$f^* = \sum_{n=0}^{\infty} a_n \varphi_n.$$

(4) $\lim\limits_{r\to 1-} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f^*(e^{it}) - f(re^{it}) \right|^2 \mathrm{d}t = 0.$
(5) $f = P[f^*]$
(6) *Let* γ *be the positively oriented unit circle. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f^*(w)}{w - z} \, \mathrm{d}w, \qquad z \in \mathbb{D}.$$

Corollary. H^2 *is a Hilbert space and the mapping* $f \mapsto f^*$ *is a linear isometry of* H^2 *onto the closed linear subspace of* $L^2(\mathbb{T})$ *generated by the functions* φ_n, $n \geq 0$. *(This subspace is formed by those* $g \in L^2(\mathbb{T})$*, whose coeficients at* φ_n, $n < 0$, *in the expansion with respect to the orthonormal basis* φ_n, $n \in \mathbb{Z}$ *vanish, i.e., by those* g *which satisfy* $\hat{g}(n) = 0$ *for* $n < 0$.*)*

Theorem 20. *Let* $p \in [1, \infty]$ *and* $f \in H^p$. *Then the limit* $f^*(e^{it}) = \lim\limits_{r\to 1-} f(re^{it})$ *exists for almost all* $t \in [0, 2\pi)$ *and, moreover, the following assertions hold.*

(1) $f^* \in L^p(\mathbb{T})$
(2) $\|f^*\|_p = \|f\|_p$
(3) $f = P[f^*]$
(4) *Let* γ *be the positively oriented unit circle. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f^*(w)}{w - z} \, \mathrm{d}w, \qquad z \in \mathbb{D}.$$

(5) *If* $p < \infty$, *then* $\lim\limits_{r\to 1-} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f^*(e^{it}) - f(re^{it}) \right|^p \mathrm{d}t = 0.$

Theorem 21. *Let* $f \in H(\mathbb{D})$ *and* $p \in [1, \infty]$. *Then* $f \in H^p$ *if and only if there exists* $g \in L^p(\mathbb{T})$ *such that* $\hat{g}(n) = 0$ *for* $n < 0$ *and* $f = P[g]$.