II.1 Analytic continuation

Definition. By a **function element** we mean an ordered pair (f, D), where $D \subset \mathbb{C}$ is an open disc and f is a holomorphic function on D. By the **center of the element** (f, D) we mean the center of the disc D.

Remark. A function element is sometimes defined in different ways – as an ordered pair (f, D), where D is an open disc, $f \in H(D)$ and, moreover, D is the disc of convergence of the Taylor series of f (centered at the center of the disc D); or as an ordered pair (f, z), where $z \in \mathbb{C}$ and f is a function holomorphic on a neighborhood of z. All the three aproaches are equivalent, but the followind definitions and statements would require a reformulation. We will use the above definition.

Definition.

- A function element (f_2, D_2) is said to be a **direct continuation** of the element (f_1, D_1) if $D_1 \cap D_2 \neq \emptyset$ and $f_1(z) = f_2(z)$ for each $z \in D_1 \cap D_2$. If the discs D_1 and D_2 have in addition the same center, we say that the elements (f_1, D_1) and (f_2, D_2) are **the same**.
- Let $\Omega \subset \mathbb{C}$ be a domain. A function element (f, D) is said to be an **analytic continuation of the element** (f_0, D_0) **in the domain** Ω if there are function elements $(f_1, D_1), \dots, (f_n, D_n)$ such that the following conditions are satisfied:
  - $D = D_n$ and $f = f_n$,
  - (f_k, D_k) is a direct continuation of (f_{k-1}, D_{k-1}) for each $k \in \{1, \dots, n\}$,
  - $D_k \subset \Omega$ for each $k \in \{0, \dots, n\}$.

Remarks:

(1) Two function elements are 'the same' if they are they are indentical in one of the above-mentioned alternative points of view. This terminology is a bit misleading but we will need it several times.

(2) The relation "to be a direct continuation of" is symmetric but not transitive. More precisely: Let (f_2, D_2) be a direct continuation of (f_1, D_1) and (f_3, D_3) be a direct continuation of (f_2, D_2). The following possibilities may occur:
  - $D_1 \cap D_3 = \emptyset$. Then (f_3, D_3) is not a direct continuation of (f_1, D_1).
  - Even if $D_1 \cap D_3 \neq \emptyset$, (f_3, D_3) need not be a direct continuation of (f_1, D_1). The functions f_1 and f_3 may differ on $D_1 \cap D_3$.
  - If $D_1 \cap D_2 \cap D_3 \neq \emptyset$, then (f_3, D_3) is a direct continuation of (f_1, D_1).

(3) The relation "to be analytic continuation in the domain Ω" is an equivalence relation.

Definition. Let (f_0, D_0) be a function element and $\gamma : [0, 1] \to \mathbb{C}$ be a continuous curve such that $\gamma(0)$ is the center of the disc D_0. Let $\Omega \subset \mathbb{C}$ be a domain.

- A function element (f, D) is said to be an **analytic continuation of** (f_0, D_0) **along** γ **in the domain** Ω if $\gamma(1)$ is the center of D and, moreover, there exist function elements $(f_1, D_1), \dots, (f_n, D_n)$ and a partion $0 = s_0 < s_1 < \dots < s_n = 1$ of the interval $[0, 1]$ such that the following conditions are satisfied:

- $D = D_n$ and $f = f_n$,
- (f_k, D_k) is a direct continuation of (f_{k-1}, D_{k-1}) for each $k \in \{1, \dots, n\}$,
- $D_k \subset \Omega$ for each $k \in \{0, \dots, n\}$,
- $\gamma([s_k, s_{k+1}]) \subset D_k$ for each $k \in \{0, \dots, n-1\}$.

- We say that the function element (f_0, D_0) **admits an analytic continuation along γ in the domain Ω**, if there is a function element (f, D) which is an analytic continuation of (f_0, D_0) alongγ in the domain Ω.

Theorem 1. *Let $\Omega \subset \mathbb{C}$ be a domain, (f_0, D_0) be a function element and $\gamma : [0, 1] \to \Omega$ be a continuous curve such that $\gamma(0)$ is the center of D_0. Then there exists at most one analytic continuation of (f_0, D_0) along γ in Ω. More precisely: If (f_1, D_1) and (f_2, D_2) are two such continuation, these two elements are the same.*

Definition. By an **analytic multifunction in the domain Ω** we mean an equivalence class of function elements with respect to the equivalence relation "to be analytic continuation in the domain Ω".

Definition. Let $\boldsymbol{f}$ be an analytic multifunction in a domain Ω.

- By the **domain of $\boldsymbol{f}$** we mean the union of all the open discs D, for which there exists $f \in H(D)$ such that $(f, D) \in \boldsymbol{f}$. The domain of $\boldsymbol{f}$ will be denoted by $\mathrm{dom}(\boldsymbol{f})$.
- Let $a \in \mathrm{dom}(\boldsymbol{f})$. By the **set of values of $\boldsymbol{f}$ at the point** a we mean the set

$$\boldsymbol{f}(a) = \{z \in \mathbb{C} : \exists (f, D) \in \boldsymbol{f} : a \in D \ \&\ f(a) = z\}.$$

- Let $G \subset \mathrm{dom}(\boldsymbol{f})$ be a domain. By a **branch of $\boldsymbol{f}$ in the domain** G we mean any analytic multifunction $\boldsymbol{g}$ in the domain G which is a subset of $\boldsymbol{f}$.

Theorem 2 (Poincaré-Volterra). *Let $\boldsymbol{f}$ be an analytic multifunction in the domain Ω and $a \in \mathrm{dom}(\boldsymbol{f})$. Then there are at most countably many function elements with center a which belong to $\boldsymbol{f}$ such that no two of them are the same. In particular, the set $\boldsymbol{f}(a)$ is at most countable.*

Definition. Let $\boldsymbol{f}$ be an analytic multifunction in the domain Ω and $p \in \mathbb{N} \cup \{\infty\}$.

- Let $a \in \mathrm{dom}(\boldsymbol{f})$. We says that $\boldsymbol{f}$ is **p-valued at the point** a, if there are exactly p function elements with center a which belong to $\boldsymbol{f}$ such that no two of them are the same (i.e., there are p such elements and in case $p \in \mathbb{N}$, there do not exist $p+1$ such elements).
- $\boldsymbol{f}$ is said to be **precisely p-valued**, if it is p-valued at each point $a \in \mathrm{dom}(\boldsymbol{f})$.
- $\boldsymbol{f}$ is said to be **p-valued** if

$$p = \sup\{q \in \mathbb{N} \cup \{\infty\} : \exists a \in \mathrm{dom}(\boldsymbol{f}), \boldsymbol{f} \text{ is } q\text{-valued at the point } a\}.$$

- 1-valued multifunctions are called **singlevalued**.

Remark. If $\boldsymbol{f}$ is p-valued at the point a, the set $\boldsymbol{f}(a)$ has at most p elements. It may have strictly less than p elements.

Remark. In case $\Omega = \mathbb{C}$, we say only **analytic continuation** instead of "analytic continuation in $\mathbb{C}$". Similarly we use the terms **analytic continuation along** γ or **analytic multifunction**.