II.3 Few facts on Riemann surfaces

Definition. By a **Riemann surface** we mean a connected holomorphic manifold of dimenson 1, i.e., a connected Hausdorff topological space X, endowed with a **holomorphic atlas**, which is a family $(U_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ with the following properties:

- U_λ is an open subset of X for each $\lambda \in \Lambda$;
- $\bigcup_{\lambda\in\Lambda} U_\lambda = X$;
- φ_λ is a homeomorphism of U_λ onto an open subset of $\mathbb{C}$ for $\lambda \in \Lambda$;
- if $\lambda, \mu \in \Lambda$ are such that $U_\lambda \cap U_\mu \neq \emptyset$, then the function $\varphi_\mu \circ \varphi_\lambda^{-1}$ is holomorphic on $\varphi_\lambda(U_\lambda \cap U_\mu)$.

The pairs $(U_\lambda, \varphi_\lambda)$ which are elements of the atlas are called **charts**. (This term is sometimes used for the maps φ_λ.)

Remarks.

(1) Any domain in $\mathbb{C}$ is a Riemann surface. (The atlas contains a single chart, the respective mapping is the identity.)
(2) $\overline{\mathbb{C}}$ is a Riemann surface. An atlas is formed by two charts $(\mathbb{C}, z \mapsto z)$ and $(\overline{\mathbb{C}} \setminus \{0\}, z \mapsto \frac{1}{z})$.
(3) Any open connected subset of a Riemann surface is again a Riemann surface (with the canonical atlas).
(4) Without loss of generality one can suppose that $\varphi(U)$ is an open disc for each chart (U, φ).

Definition. Let X and Y be Riemann surfaces and $\Omega \subset X$ be an open subset.

- A mapping $f : \Omega \to \mathbb{C}$ is said to be **holomorphic** if for any chart (U, φ) such that $U \cap \Omega \neq \emptyset$ the function $f \circ \varphi^{-1}$ is holomorphic on $\varphi(U \cap \Omega)$.
- A mapping $f : \Omega \to Y$ is said to be **holomorphic** if it is continuous and for each chart (U, φ) on X and each chart (V, ψ) on Y the mapping $\psi \circ f \circ \varphi^{-1}$ is holomorphic on $\varphi(U \cap f^{-1}(V))$ (whenever this set is nonempty).
- A mapping $f : \Omega \to \overline{\mathbb{C}}$ is said to be **meromorphic** if for any chart (U, φ) such that $U \cap \Omega \neq \emptyset$ the function $f \circ \varphi^{-1}$ is meromorphic on $\varphi(U \cap \Omega)$.

Theorem 7 (properties of holomorphic mappings between Riemann surfaces).

(1) *Let X be a Riemann surface and f, g be holomorphic mappings of X to $\mathbb{C}$. Then the functions $f + g$ and fg are holomorphic as well. If g does not attain zero, the function f/g is holomorphic as well.*
(2) *Let X, Y and Z be Riemann surfaces, $f : X \to Y$ and $g : Y \to Z$ be holomorphic mappings. Then $g \circ f$ is holomorphic as well.*
(3) *Let X and Y be Riemann surfaces. Let $f : X \to Y$ and $g : X \to Y$ be holomorphic mappings. If the set $\{x \in X : f(x) = g(x)\}$ has an accumulation point in X, then $f = g$ on X.*
(4) *Let X be a Riemann surface and $f : X \to \mathbb{C}$ be a non-constant holomorphic mapping. Then $|f|$ does not attain local maximum at any point of X.*

Corollary. *Let X be a compact Riemann surface. Then any holomorphic function $f : X \to \mathbb{C}$ is constant.*

Remark:

(1) Any Riemann surface admits a nonconstant meromorphic function.
(2) Any noncompact Riemann surface admits a nonconstant holomorphic function.

Riemann surface of an analytic multifunction

Let $\boldsymbol{f}$ be an analytic multifunction in the domain Ω. Then there is a Riemann surface and a holomorphic function corresponding to $\boldsymbol{f}$. It can be described as follows:

- The set X is the set of all the equivalence classes with respect to the equivalence relation "to be the same element as" defined on $\boldsymbol{f}$.
- The topology of X: Let $\boldsymbol{x} \in X$. Let $(f, D) \in \boldsymbol{x}$ and z_0 is the center of D. For any $r > 0$ set

$$\begin{aligned} U(\boldsymbol{x}, r) = \{\boldsymbol{y} \in X : \exists (g, U(z, \delta)) \in \boldsymbol{y} : |z - z_0| < r \\ \& \ (g, U(z, \delta)) \text{ is a direct continuation of } (f, D)\}. \end{aligned}$$

  The family $U(\boldsymbol{x}, r)$, $r > 0$ then forms a neighborhood basis of $\boldsymbol{x}$ in X.
- The atlas on X consists of charts $(U(\boldsymbol{x}, r), \varphi)$ for $\boldsymbol{x} \in X$ and $r > 0$, where

$$\varphi : \boldsymbol{y} \mapsto \text{ the center of } D \text{ where } (D, f) \in \boldsymbol{y}.$$

Further, let us define a function $F : X \to \mathbb{C}$ by the formula:

$$F(\boldsymbol{x}) = f(z), \text{ if } (f, D) \in \boldsymbol{x} \text{ and } z \text{ is the center of } D.$$

Then F is holomorphic on X.