III.2 Holomorphic functions of several complex variables

Definition. Let $\Omega \subset \mathbb{C}^n$ be an open set and $f : \Omega \to \mathbb{C}$ be a function. The function f is said to be

- **holomorphic on** Ω, if for each $\boldsymbol{x} \in \Omega$ there exist coefficients c_α, $\alpha \in \mathbb{N}_0^n$ such that

$$f(\boldsymbol{y}) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha (\boldsymbol{y} - \boldsymbol{x})^\alpha$$

  for $\boldsymbol{y}$ from a neighborhood of $\boldsymbol{x}$.
- **separately holomorphic on** Ω, if for each $j \in \{1, \dots, n\}$ and for any choice of $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n \in \mathbb{C}$ the function

$$z \mapsto f(x_1, \dots, x_{j-1}, z, x_{j+1}, \dots, x_n)$$

  is holomorphic on its domain.

Remark. A function f is separately holomorphic on Ω, if and only if it admits finite partial derivatives with respect to all variables at any point of Ω.

Definition. Let $X_1, \dots, X_n$ and Y be metric spaces. A mapping $f : X_1 \times \cdots \times X_n \to Y$ is said to be **separately continuous**, if for each $j \in \{1, \dots, n\}$ and for any choice of $x_k \in X_k$, $k \in \{1, \dots, n\} \setminus \{j\}$ the mapping

$$x \mapsto f(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_n)$$

is continuous on X_j.

Lemma 6.

(1) *Let X, Y, Z be metric spaces such that X is separable. Let $f : X \times Y \to Z$ satisfy the following two conditions.*
   - *For each $y \in Y$ the mapping $x \mapsto f(x, y)$ is continuous on X.*
   - *For each $x \in X$ the mapping $y \mapsto f(x, y)$ is Borel-measurable on Y.*

   Then f is Borel measurable on $X \times Y$.

(2) *Let $X_1, \dots, X_n$ be separable metric spaces, Z a metric space and $f : X_1 \times \cdots \times X_n \to Z$ a separately continuous mapping. Then f is Borel measurable.*

Notation: Let $\boldsymbol{x} \in \mathbb{C}^n$ and $\boldsymbol{r} \in (0, +\infty)^n$. Then we set

$$\mathbb{P}(\boldsymbol{x}, \boldsymbol{r}) = \prod_{j=1}^{n} U(x_j, r_j).$$

A set of this form is said to be a **polydisc**.

Lemma 7. *Let f be separately holomorphic on an open set $\Omega \subset \mathbb{C}^n$. Let $\boldsymbol{x} \in \Omega$ and $\boldsymbol{r} \in (0, +\infty)^n$ be such that $\overline{\mathbb{P}(\boldsymbol{x}, \boldsymbol{r})} \subset \Omega$. Then for each $\boldsymbol{y} \in \mathbb{P}(\boldsymbol{x}, \boldsymbol{r})$ the following equality holds:*

$$f(\boldsymbol{y}) = \frac{1}{(2\pi i)^n} \int_0^{2\pi} \left(\cdots \int_0^{2\pi} \left(\int_0^{2\pi} \frac{f(x_1 + r_1 e^{it_1}, x_2 + r_2 e^{it_2}, \dots, x_n + r_n e^{it_n})}{(x_1 + r_1 e^{it_1} - y_1)(x_2 + r_2 e^{it_2} - y_2) \cdots (x_n + r_n e^{it_n} - y_n)} \right.\right.$$
$$\left.\left. \cdot\, r_1 \cdots r_n \cdot i^n \cdot e^{i(t_1 + \dots + t_n)} \, \mathrm{d}t_1 \right) \mathrm{d}t_2 \dots \right) \mathrm{d}t_n$$

Theorem 8. *Let $\Omega \subset \mathbb{C}^n$ be an open set and $f : \Omega \to \mathbb{C}$ be a function. The following assertions are equivalent.*

1. *f is holomorphic on Ω.*
2. *f admits a Fréchet derivative at each point of Ω.*
3. *f is separately holomorphic and locally bounded on Ω.*
4. *Whenever $\boldsymbol{x} \in \Omega$ and $\boldsymbol{r} \in (0, +\infty)^n$ are such that $\overline{\mathbb{P}(\boldsymbol{x}, \boldsymbol{r})} \subset \Omega$, then f is bounded on $\prod_{j=1}^{n} \{z \in \mathbb{C} : |z - x_j| = r_j\}$ and for each $\boldsymbol{y} \in \mathbb{P}(\boldsymbol{x}, \boldsymbol{r})$ the following formula holds:*

$$f(\boldsymbol{y}) = \frac{1}{(2\pi i)^n} \int_{[0,2\pi]^n} \frac{f(x_1 + r_1 e^{it_1}, x_2 + r_2 e^{it_2}, \dots, x_n + r_n e^{it_n})}{(x_1 + r_1 e^{it_1} - y_1)(x_2 + r_2 e^{it_2} - y_2) \cdots (x_n + r_n e^{it_n} - y_n)}$$
$$\cdot\, r_1 \cdots r_n \cdot i^n \cdot e^{i(t_1 + \dots + t_n)} \, \mathrm{d}t_1 \, \mathrm{d}t_2 \dots \mathrm{d}t_n$$

5. *For each $\boldsymbol{x} \in \Omega$ there is $\boldsymbol{r} \in (0, +\infty)^n$ such that $\overline{\mathbb{P}(\boldsymbol{x}, \boldsymbol{r})} \subset \Omega$ and the conclusion of the previous assertion holds.*

Remark. It even holds that any separately holomorphic function is holomorphic. This is the content of Hartogs theorem which will be addressed later.

Theorem 9. *Let f be a holomorphic function on an open set $\Omega \subset \mathbb{C}^n$. Then for each $\alpha \in \mathbb{N}_0^n$ the function $\frac{\partial^{|\alpha|}}{\partial \boldsymbol{x}^\alpha} f$ is holomorphic on Ω.*

Theorem 10. *Let $\Omega \subset \mathbb{C}^n$ be an open set and (f_n) be a sequence of holomorphic functions on Ω, which converges to a function f locally uniformly on Ω. Then f is holomorphic on Ω as well and, moreover, for each $\alpha \in \mathbb{N}_0^n$ the functions $\frac{\partial^{|\alpha|}}{\partial \boldsymbol{x}^\alpha} f_n$ converge to $\frac{\partial^{|\alpha|}}{\partial \boldsymbol{x}^\alpha} f$ locally uniformly on Ω.*