III.3 Hartogs extension theorem and domains of holomorphy

Proposition 11. *Let $n \in \mathbb{N}$, $n \geq 2$. Let $G \subset \mathbb{C}^{n-1}$ be a domain, $z \in \mathbb{C}$ and $r > 0$. Set $\Omega = G \times U(z,r)$. Let $V \subset \Omega$ be a domain satisfying the following two conditions:*

- *There exists $s \in (0,r)$ such that $\Omega \setminus V \subset G \times U(z,s)$.*
- *There exists a nonempty open set $H \subset G$ such that $H \times U(z,r) \subset V$.*

Then any holomorphic function on V can be extended to a holomorphic function on Ω.

Lemma 12. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\Omega \subset \mathbb{C}^n$ be a domain. Denote by p the projection of $\mathbb{C}^n$ onto $\mathbb{C}^{n-1}$ defined by deleting the last coordinate. Then for any compact set $K \subset \Omega$ and any $\boldsymbol{x} \in p(\Omega)$ there exists a cycle Γ in $\mathbb{C}$ and a polydisc $\mathbb{P}(\boldsymbol{x}, \boldsymbol{r}) \subset p(\Omega)$ satisfying the following properties:*

(i) $\forall \boldsymbol{y} \in \mathbb{P}(\boldsymbol{x}, \boldsymbol{r})\ \forall z \in \mathbb{C} : (\boldsymbol{y}, z) \in \mathbb{C}^n \setminus \Omega \Rightarrow \operatorname{ind}_\Gamma z = 0.$

(ii) $\forall \boldsymbol{y} \in \mathbb{P}(\boldsymbol{x}, \boldsymbol{r})\ \forall z \in \mathbb{C} : (\boldsymbol{y}, z) \in K \Rightarrow \operatorname{ind}_\Gamma z = 1.$

Theorem 13 (Hartogs extension theorem). *Let $n \in \mathbb{N}$, $n \geq 2$. Let $\Omega \subset \mathbb{C}^n$ be a domain and $K \subset \Omega$ be a compact set such that $\Omega \setminus K$ is connected. Then any holomorphic function on $\Omega \setminus K$ can be extended to a holomorphic function on Ω.*

Definition. A domain $\Omega \subset \mathbb{C}^n$ is said to be a **domain of holomorphy**, provided there exists a function f holomorphic on Ω such that whenever $\boldsymbol{x}$ is a boundary point of Ω and $\mathbb{P}(\boldsymbol{x}, \boldsymbol{r})$ is an arbitrary polydisc centered at $\boldsymbol{x}$, then there does not exist any function g holomorphic on $\mathbb{P}(\boldsymbol{x}, \boldsymbol{r})$, which agrees with f on at least one component of $\Omega \cap \mathbb{P}(\boldsymbol{x}, \boldsymbol{r})$.

Remark: Any nonempty domain in $\mathbb{C}$ is a domain of holomorphy.

Theorem 14. *A domain $\Omega \subset \mathbb{C}^n$ is a domain of holomorphy if and only if for any compact set $K \subset \Omega$ the set*

$$\hat{K} = \{\boldsymbol{x} \in \Omega : \forall f : \Omega \to \mathbb{C} \text{ holomorphic } |f(\boldsymbol{x})| \leq \sup_{\boldsymbol{y} \in K} |f(\boldsymbol{y})|\}$$

is again compact.

Corollary. *Any nonempty convex open set in $\mathbb{C}^n$ is a domain of holomorphy.*