General comments: I followed the same rules as in the previous tests – to give a partial credit for a reasonable part of the solution and to require at least one half of the points (i.e., at least 1.5 points). Then five students passed the test and eight students failed. (With the original strict rules, only one student would pass.)

Overall statistics: Three students have passed two tests, six students passed one test; remaining students (five) have not passed any test yet.

Problem 1

Solution: We need to find all $x \in \mathbb{R}$ such that

$$\operatorname{arccotg}(x^2) < \frac{\pi}{3}.$$

The domain of arccotg is whole real line, arccotg is strictly decreasing on $\mathbf{R}$ and $\frac{\pi}{3} = \operatorname{arccotg} \frac{1}{\sqrt{3}}$. Therefore the above inequality is equivalent to

$$x^2 > \frac{1}{\sqrt{3}},$$

i.e.,

$$|x| > \frac{1}{\sqrt[4]{3}}.$$

So, the solution set is $(-\infty, -\frac{1}{\sqrt[4]{3}}) \cup (\frac{1}{\sqrt[4]{3}}, +\infty)$.

Evaluation:

Two students provied a complete correct solution. They obtained 1 point.

Six students tried to follow an essentially correct direction. One of them wrote that arccotg is decreasing but did not reverse the respective inequality; another one correctly reversed the inequality, but mistook $\sqrt{3}$ for $\frac{1}{\sqrt{3}}$ and later 3 for $\sqrt{3}$. These two students got 0.6 points. Three students did not reverse the inequality (one of them remarked that arccotg is decreasing, the other two not) and, moreover, did not correctly solved the inequality $x^2 < \frac{1}{\sqrt{3}}$. They got 0.3 points. One students mistook $\frac{1}{2}$ for $\frac{1}{\sqrt{3}}$, did not reverse the inequality and, moreover, did not correctly solved the inequality $x^2 < \frac{1}{2}$. This one obtained 0.2 points.

One student probably confused arccotg with arctg and, moreover, claimed that the domain of arctg is $\langle -\sqrt{3}, \sqrt{3} \rangle$ (which is a nosense). However, some traces of the correct way were present, so the result was 0.2 points.

The remaining four students got 0 points. One of them did not include any computation, three of them made some computation which is a complete nonsense.

Overall comments on this problem: The problem was very similar to the first problems in Test 1 and Test 2. In fact, it was even easier as it is clear from the solution. However, 7 students got in all three tests $0 - 0.4$ points from this problem. I hope they will do better next time.

Problem 2:

Solution: The integer part $[x]$ satisfies the inequality $x-1<[x]\le x$. Since the function log is increasing and the denominator is positive, we have

$$\frac{\log(\sqrt[5]{n}+6)}{\log(n^3+2)}<\frac{\log([\sqrt[5]{n}]+7)}{\log(n^3+2)}\le\frac{\log(\sqrt[5]{n}+7)}{\log(n^3+2)}$$

Further,

$$\lim_{n\to\infty}\frac{\log(\sqrt[5]{n}+7)}{\log(n^3+2)}\overset{(1)}{=}\lim_{n\to\infty}\frac{\log(\sqrt[5]{n})+\log(1+\frac{7}{\sqrt[5]{n}})}{\log(n^3)+\log(1+\frac{2}{n^3})}\overset{(2)}{=}\lim_{n\to\infty}\frac{\frac{1}{5}\log n+\log(1+\frac{7}{\sqrt[5]{n}})}{3\log n+\log(1+\frac{2}{n^3})}$$

$$\overset{(3)}{=}\lim_{n\to\infty}\frac{\frac{1}{5}+\frac{\log(1+\frac{7}{\sqrt[5]{n}})}{\log n}}{3+\frac{\log(1+\frac{2}{n^3})}{\log n}}\overset{(4)}{=}\frac{\frac{1}{5}}{3}=\frac{1}{15}.$$

Steps (1) and (2) use the properties of the function log (namely the formula $\log(xy)=\log x+\log y$ for $x,y>0$. In the step (3) we divided both nominator and denominator by the leading term. Step (4) was done using the "aritmetic of limits", since the nominator has limit $\frac{1}{5}$ and the denominator has limit 3.

In the same way we can compute that

$$\lim_{n\to\infty}\frac{\log(\sqrt[5]{n}+6)}{\log(n^3+2)}=\frac{1}{15}$$

as well, so by the sandwich theorem the original limit equals $\frac{1}{15}$.

Evaluation: Five students got 0 points. Two of them made no computation at all. One of them mentioned sandwich theorem, but the estimates were completely useless, the remaining two performed some strange steps (one of them claimed that the limit of a function equals to the limit of its derivative, which is a complete nonsense).

Three students tried to use the sandwich theorem in a correct way. One of them continued in a good direction with some mistakes (for example, the constant 7 diseappeared from the nominator, the order of steps was not completely correct) and got 0.7 points. The other two then have not really continued. They got 0.2 points.

One student followed the correct way, but did not use the proper steps, but rather a good intuition. There were some more precise steps written but crossed off by the student. The result was 0.6 points.

The remaining four student have not treated the integer part. Two of them simply computed pretending that the integer part is not present here and then used an essentially good direction but bad order of steps. They got 0.4 points. The other two tried to use l'Hospital rule (which is strange for a sequence and even more strange for a sequence containing the integer part) and arrived to nothing. They got 0.2 points.

Problem 3:

Solution: The solution consists in the following parts - determining the domain of g, drawing a picture of the domain, computing the formulas for partial derivatives and observing that those formulas are valid on the whole domain of g.

The function g can be expressed by

$$g(x, y) = \exp(\operatorname{arctg}(xy) \cdot \log(y - \sin x)),$$

the domain of g is described by the condition $y - \sin x > 0$. (in order $\log(y - \sin x)$ is defined. Hence the domain of g is equal to $\{[x, y] \in \mathbb{R}^2 : y > \sin x\}$, which is the area above the graph of sin.

Partial derivatives can be computed in the standard way:

$$\frac{\partial g}{\partial x}(x, y) = \exp(\operatorname{arctg}(xy) \cdot \log(y - \sin x)) \cdot \left(\frac{1}{1 + (xy)^2} \cdot y \cdot \log(y - \sin x) + \operatorname{arctg}(xy) \cdot \frac{1}{y - \sin x} \cdot (-\cos x) \right),$$
$$\frac{\partial g}{\partial y}(x, y) = \exp(\operatorname{arctg}(xy) \cdot \log(y - \sin x)) \cdot \left(\frac{1}{1 + (xy)^2} \cdot x \cdot \log(y - \sin x) + \operatorname{arctg}(xy) \cdot \frac{1}{y - \sin x} \right).$$

It is also clear that these formulas are valid on the whole domain of g. The formulas can be simplified a bit.

Evaluation:

Two students provided a complete correct solution and obtained 1 point.

Another two students provided a complete almost correct solution with only a small mistake – one of them got 0.9 points, the other one 0.8.

Two students correctly computed the partial derivatives (one of them with a small mistake, maybe a misprint), but their domains were not correct. One of them claimed that the domain is whole plane, the other one got some strange inequalities. Both of them obtained 0.6 points.

Three students correctly determined the domain, made no picture and the partial derivatives computed in a bad way. They got 0.2 points.

The remaining four students got 0 points as I found nothing correct in their solution.