Mathematics in Juggling
Juggling in Mathematics

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The Jarní doktorandská škola didaktiky matematiky
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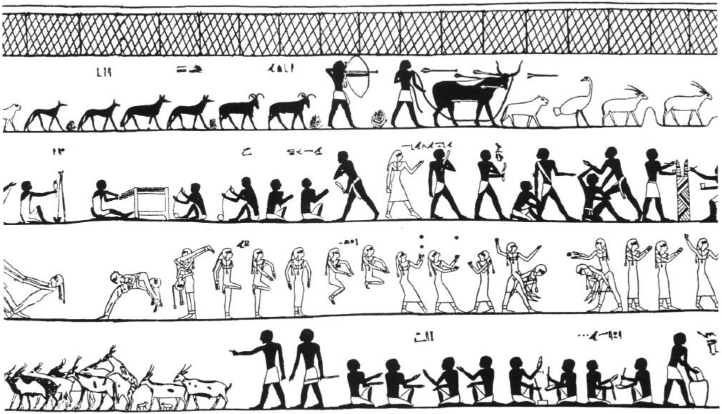
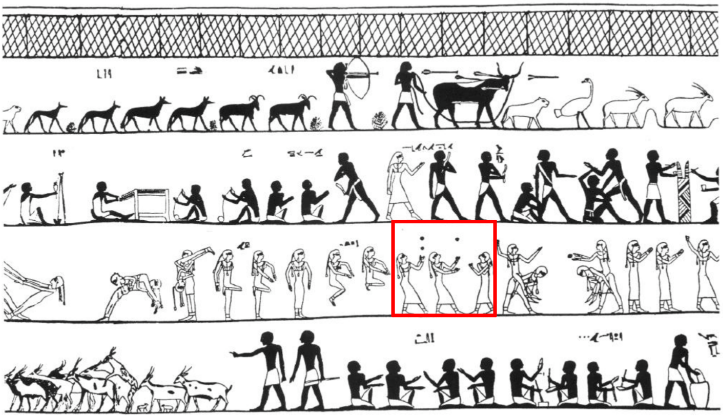


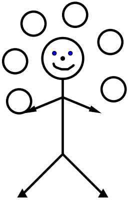
The first historical evidence of juggling in the Beni Hassan location in Egypt, between 1994-1781 B.C..

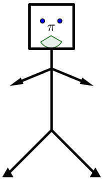


The first historical evidence of juggling in the Beni Hassan location in Egypt, between 1994-1781 B.C..

- Why? ... describe juggling
- How? ... describe juggling

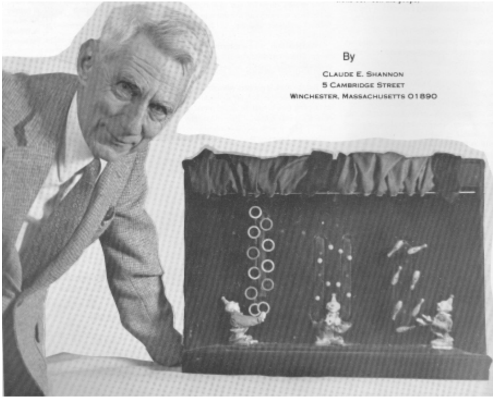
Why?





JUGGLER	MATHEMATICIAN
- "language"	- notation
- new tricks	- new properties
- understanding principles	- application to related theories

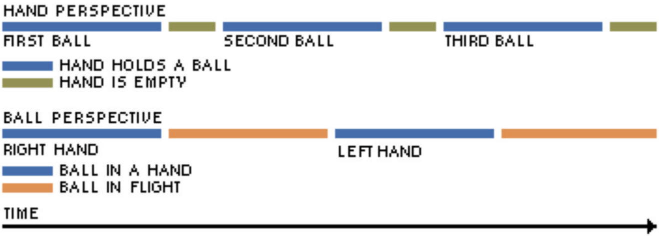
Claude Elwood Shannon, 1916-2001



- Construction of a juggling robot (1970s)
- Underlying mathematical concept - Uniform juggling

Claude Elwood Shannon, 1916-2001

- h hands, b balls
  - d Dwell time ball, or occupation time hand
  - f Flight time of a ball
  - e Empty hand time



Theorem (Shannon 1st juggling theorem.)

In the uniform juggling, it holds:

$$\frac{f+d}{e+d}=\frac{b}{h}$$

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Duality principle. We can exchange the terms ball and hand (and related terms).

$d = 0$ minimal frequency of juggling
$e = 0$ maximal frequency of juggling
f flight time is constant (thus, also the height of throw)

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Theorem (Frequency of the uniform juggling)

The ratio of maximal and minimal frequency in uniform juggling is

$$\frac{b}{b-h}.$$

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Theorem (Frequency of the uniform juggling)

The ratio of maximal and minimal frequency in uniform juggling is

$$\frac{b}{b-h}.$$

Proof.

$$d = \frac{fh - eb}{b - h} \qquad (1)$$

$$e = \frac{(d+f)h}{b} - d \qquad (2)$$

maximal frequency, $e = 0$ *in (1):* $d = \frac{fh}{b-h}$

minimal frequency, $d = 0$ *in (2):* $e = \frac{fh}{b}$

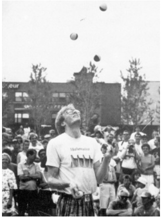
ratio of max and min frequency is $\frac{d}{e} = \frac{b}{b-h}$

□

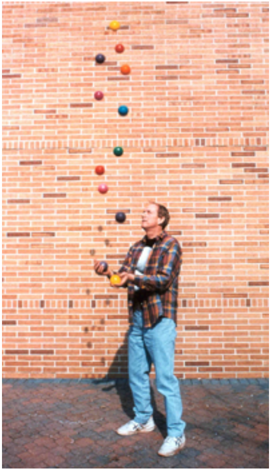
Examples:

human_1	cascade with 3 balls and 2 hands gives $\frac{3}{1}$
human_2	fountain with 4 balls and 2 hands gives $\frac{2}{1}$
still human	cascade 7 balls for 2 hands gives ratio $\frac{7}{5}$
robot	$2n+1$ balls and 2 hands $\frac{2n+1}{2n-1}$
passing jugglers	11 balls 4 hands $\frac{11}{7}$

- cca 1985 - 2 independent groups found a juggling notation with the use of integer sequences
- Paul Klimak from Santa Cruz, Bent Magnusson and Bruce "Boppo" Tiemann from Los Angeles - Caltech, USA
- Adam Chalcraft, Mike Day and Colin Wright from Cambridge, UK



Ronald Graham, 1935



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- A juggler:

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  - stands at one place
  - hands are in fixed positions
- Simplifying throws (propositions)

- A juggler:
  - stands at one place
  - hands are in fixed positions
- Simplifying throws (propositions)
- A juggler:
  1) throws the balls on constant beats
  2) has always been juggling and will never end
  3) throws on each beat at most one ball, and if he catches some ball, he must throw it

- Divide juggling into separate throws
- *Throw* = movement of the ball since it was thrown until it landed
- *Height of a throw* = number of beats which pass since the ball was thrown until it landed (including landing)

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- Juggler has (usually) two hands
    - odd throws land into the other hand
    - even throws land into the same hand
- Juggling the same throw on each beat:
    - odd $\rightarrow$ *cascade*
    - even $\rightarrow$ *fountain*

- *Juggling function* ϕ:
  assigns the height to each throwing time (beat)
  $\phi : \mathbb{Z} \to \mathbb{N}_0$
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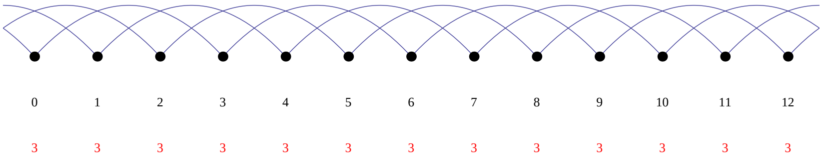
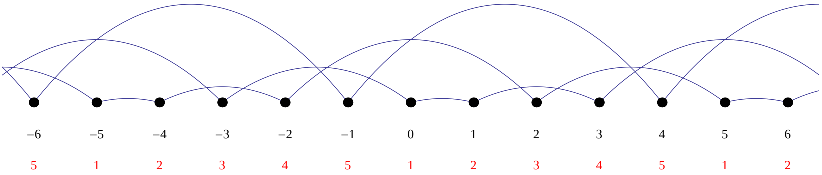
- Function is said to be *("simple") juggling*, if its landing function is permutation of integers

$$\phi(i) = \ldots 3333 \ldots$$
$$\overline{\phi(i)} = \ldots 3456 \ldots$$



i	...	−3	−2	−1	0	1	2	3	4	5	6	...
$\phi(i)$	...	3	4	5	1	2	3	4	5	1	2	...
$\overline{\phi(i)}$	...	0	2	4	1	3	5	7	9	6	8	...



- *Trick* - repeating pattern in juggling
- *Juggling sequence* (Siteswap) $\{h_k\}$:
  $\{h_k\}_{k=1}^{p} \dots$ is finite sequence of heights of throws $(\mathbb{N}_0)$
  $\phi(i) = h_{i \mod p}, \forall i \in \mathbb{Z}$
- $\phi(i)$ ("simple) juggling function $\implies$ $\{h_k\}$ is said to be *"simple" juggling sequence* or *siteswap of a length p*

$$h_1 h_2 \dots h_p$$

- Examples of siteswaps:
  33333, 3 (cascade), 441441, 12345, 7531, 97531, 88441

Average test

Theorem (Average theorem (necessary condition))

The number of balls necessary to juggle a juggling sequence $\{h_k\}_{k=1}^{p}$ equals its average $\frac{\sum_{k=0}^{p-1} h_k}{p}$.

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- Siteswap 12345 contains $\frac{1+2+3+4+5}{5} = 3$ balls

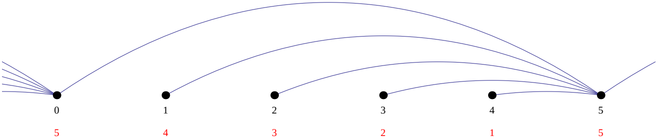
Average test

Theorem (Average theorem (necessary condition))

The number of balls necessary to juggle a juggling sequence $\{h_k\}_{k=1}^{p}$ equals its average $\frac{\sum_{k=0}^{p-1} h_k}{p}$.

- Siteswap 12345 contains $\frac{1+2+3+4+5}{5} = 3$ balls
- Reverse theorem does not hold in general

- Siteswap 54321 holds the condition of integer average, but it is in contradiction with the properties of juggling.



- The conversed theorem in the following manner holds:

Theorem („Conversed“ average theorem)

Let us have a set of nonnegative integers with integer average, then we can rearrange them to a juggling sequence.

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- Let us have throws of heights 3,3,5,6,8.

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Theorem („Conversed“ average theorem)

Let us have a set of nonnegative integers with integer average, then we can rearrange them to a juggling sequence.

- Let us have throws of heights 3,3,5,6,8.
- Rearranged sequence is 85363.

Permutation test

- Permutation test - generator
- The *generator of a juggling sequence* is the sequence $\{h_k \mod p\}_{k=0}^{p-1}$

Theorem

The generator of a juggling sequence is a juggling sequence.

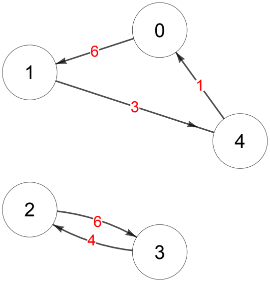
$[63641] \mod 5 \to 13141$

- Landing times of balls makes permutation:

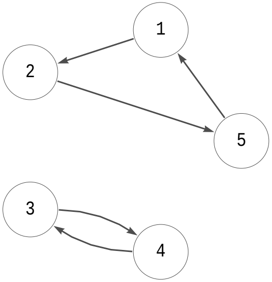
height of throw	6	3	6	4	1
time	0	1	2	3	4
landing time	6	4	8	7	5
land mod 5	1	4	3	2	0

- sufficient condition

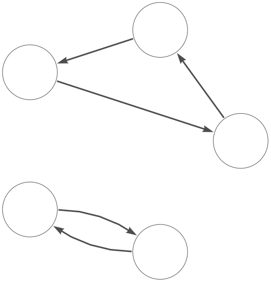
- Graphical representation of a siteswap with the cyclic diagram



- 63641
- vertex $\leftrightarrow$ time, edge $\leftrightarrow$ throw
- In each vertex starts and finishes exactly one oriented edge
- Holds the properties of our model



The cyclic diagram
of the siteswap 63641



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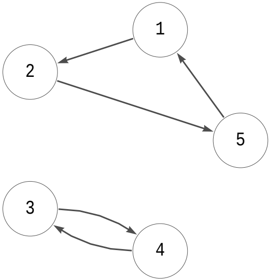

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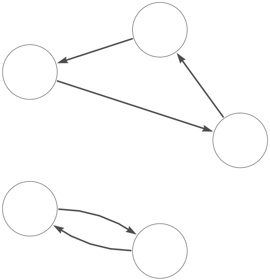
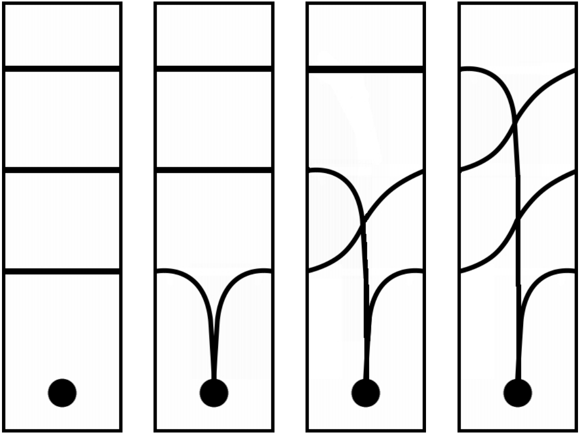


The generator
of the siteswap 63641

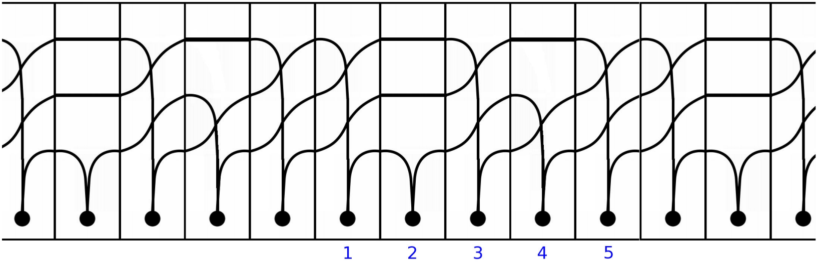
- Method of constructing new siteswaps with the use of generator - drawing diagram

Juggling cards



Juggling cards K_0, K_1, K_2 a K_3 for 3 balls

Juggling cards



The siteswap 12345 with juggling cards

Theorem

The number of all juggling sequences of the period p with at most b balls is:

$$S(b,p) = (b+1)^p$$

Theorem

The number of all juggling sequences of the period p with b balls is:

$$\overline{S(b,p)} = S(b,p) - S(b-1,p) = (b+1)^p - b^p$$

Looking for siteswaps without repetitions (e.g. 737373)

Theorem

The number of all minimal juggling sequences of the period p with b balls without cyclic shifts is:

$$MS(b,p) = \frac{1}{p}\sum_{d|p}\mu(\frac{p}{d})((b+1)^d - b^d)$$

Number of juggling sequences

- The number of all minimal juggling sequences of the period p with b balls without cyclic shift is:

$$MS(b,p) = \frac{1}{p}\sum_{d|p} \mu(\frac{p}{d})((b+1)^d - b^d)$$

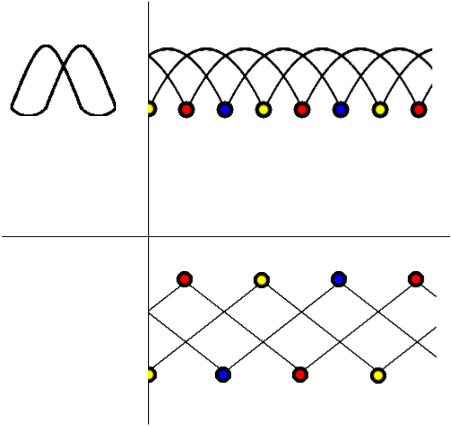
- The number of all generators of a juggling sequences of the period p is:

$$G(p) = \frac{1}{p}\sum_{d|p} \varphi\left(\frac{p}{d}\right)\left(\frac{p}{d}\right)^d d!$$

- The number of generators of the period 60 is:

138 683 118 545 689 835 737 939 019 720 389 406 345 907 623 657 512 698
795 667 111 474 180 725 129 470 672.

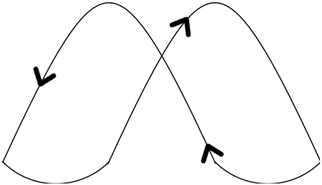
Projections of trajectories of balls

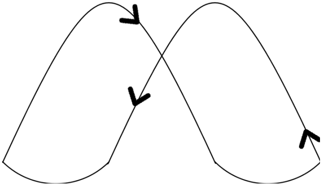


- Model of juggling - hands are in fixed positions $\Longrightarrow$ balls will collide
- Extension of the model, to characterize the created braid

Cascade and Reverse cascade

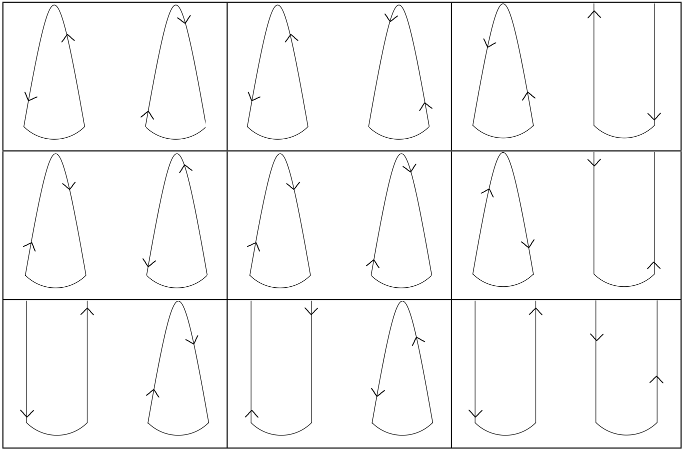
- Inside and outside throws in 3-cascade





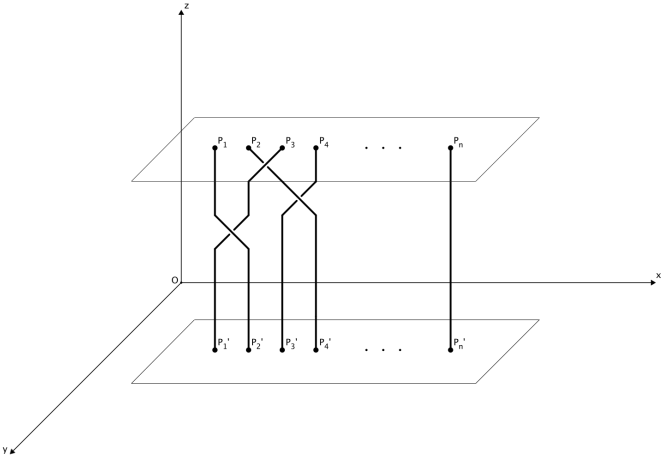
Fountains

- Inside and outside throws in 4-fountains



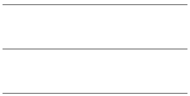
Theory of braids

- Space model of a braid

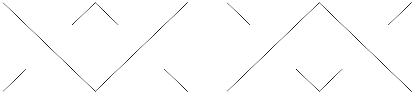


Braid diagram

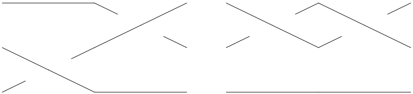
Trivial braid



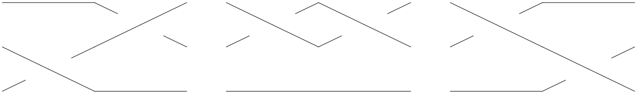
- Braids can be continuously deformed



Two equivalent braids

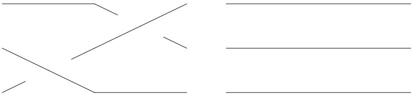


Composition of two braids

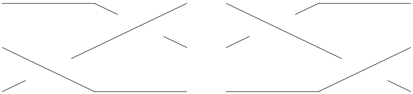


Associativity of composition:

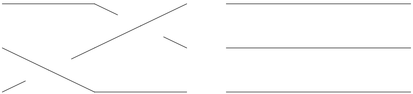
$$\alpha(\beta\gamma) = (\alpha\beta)\gamma$$



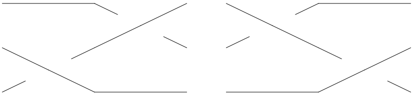
Composition of a braid α with the trivial braid ϵ



Composition of braids $\alpha\alpha^{-1}$ is trivial braid

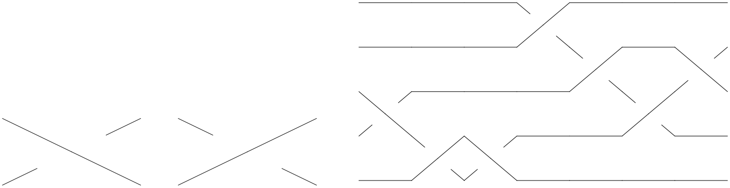


Composition of a braid α with the trivial braid ϵ



Composition of braids $\alpha\alpha^{-1}$ is trivial braid

- Braids make the Braid group



Braid generators and braid words: $\sigma_3\sigma_4^{-1}\sigma_4\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_3$

- With braids we can describe juggling with respect to inside and outside throws
- In siteswap, the time between a catch and throw is zero $\Longrightarrow$ we need a "mathematical" description of the rules of movement of a ball with the use of the inside and outside throws

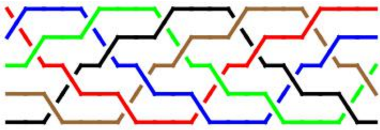
- With braids we can describe juggling with respect to inside and outside throws
- In siteswap, the time between a catch and throw is zero $\Longrightarrow$ we need a "mathematical" description of the rules of movement of a ball with the use of the inside and outside throws

(i) The ball thrown at present by an inside (outside) throw will pass under (above) all the balls, which were thrown earlier and will land earlier than the given ball, if all considered balls will land into the same hand from which we are throwing.

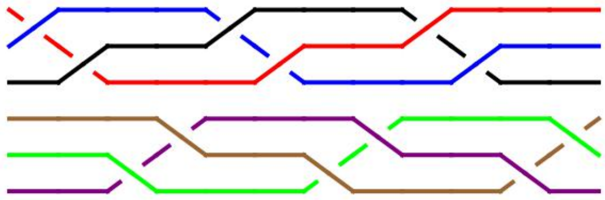
(ii) The ball thrown at present by an inside (outside) throw of an odd height will pass under all the balls, which were thrown earlier and will land later than the given ball.

(iii) The ball thrown at present by an inside (outside) throw of an even height will pass under all the balls, which were thrown earlier and will land later than the given ball, if all considered balls will land into the same hand from which we are throwing.

Braid words and diagrams of cascades and fountains.



Cascade with 5 balls = $\sigma_1^{-1}\sigma_2^{-1}\sigma_4\sigma_3 \ldots \sigma_1^{-1}\sigma_2^{-1}\sigma_4\sigma_3$



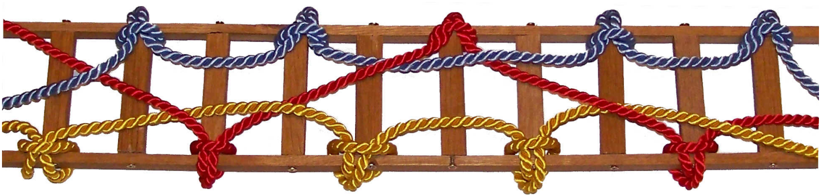
Fountain with 6 balls = $\sigma_1^{-1}\sigma_2^{-1}\sigma_5\sigma_4 \ldots \sigma_1^{-1}\sigma_2^{-1}\sigma_5\sigma_4$

Braid words and diagrams of cascades and fountains.

$2n$ balls $\sigma_1^{-1}\sigma_2^{-1}\dots\sigma_{n-1}^{-1}\sigma_{2n-1}\sigma_{2n-2}\dots\sigma_{n+1}$ n-times

$2n+1$ balls $\sigma_1^{-1}\sigma_2^{-1}\dots\sigma_n^{-1}\sigma_{2n}\sigma_{2n-1}\dots\sigma_{n+1}$ $(2n+1)$-times

Juggling braids of siteswaps



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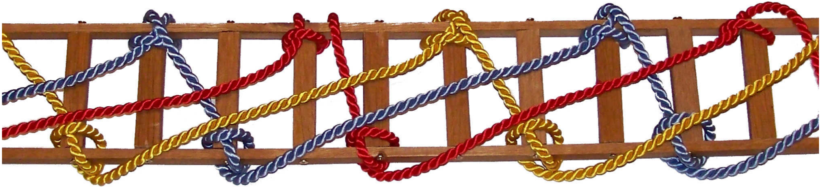
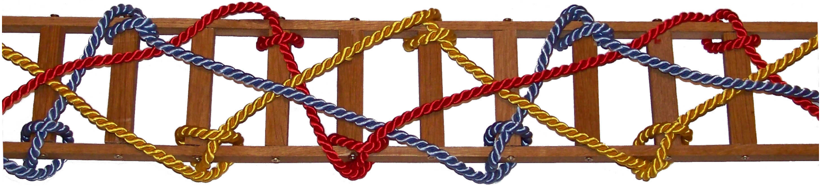
Juggling braids of siteswaps



51

both throws can be inside or outside

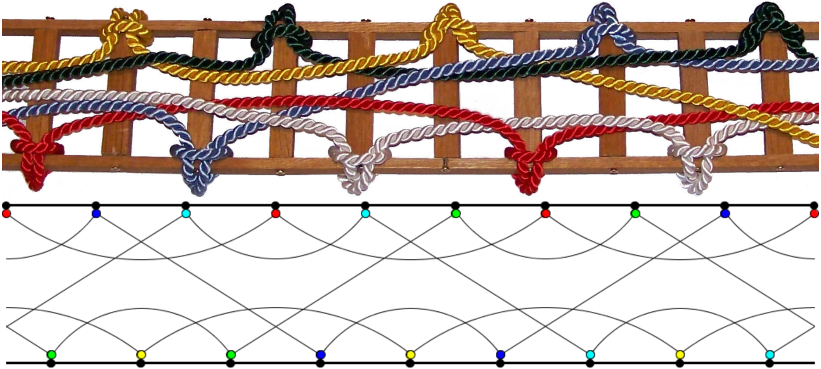
Juggling braids of siteswaps



531

all balls lie on a vertical line at some points in time

Juggling braids of siteswaps



645

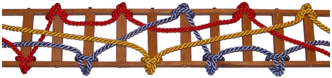
to compose braids of juggling tricks we need them to finish in the starting position

Juggling braids of siteswaps

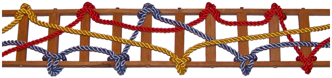
cascade IIO as a trivial braid

choice of inside and outside throws is arbitrary

- Inverse braid (reverse siteswap) unbraids the original braid



Siteswap 12345 with inside throws



Siteswap 52413 outside throws

"Each braid is juggleable"

- Creating a siteswap of an arbitrary braid.
  (i) siteswap of trivial braid
  (ii) siteswaps of braid generators
  (iii) composing the final siteswap of the braid
- Using a different model of inside/ outside throws.
- The final siteswap is impossible to juggle

Thank you for your attention!