12 Orders

12.1. Prove that any group of prime order is cyclic.

Hint: Using the Lagrange theorem show that the order of an arbitrary non-trivial cyclic subgroup is of the same order as the group.

12.2. In the group $\mathbf{S}_5$ determine the order of the cyclic subgroup $\langle \pi \rangle_{\mathbf{S}_5}$ and the index $[\mathbf{S}_5 : \langle \pi \rangle_{\mathbf{S}_5}]$ if

(a) $\pi = (1\,2\,3\,4\,5)$,

(b) $\pi = (1\,2)(3\,4\,5)$,

(c) $\pi = \mathrm{id}$.

Solutions: (a) $|\langle \pi \rangle| = 5$, $[\mathbf{S}_5 : \langle \pi \rangle] = 4! = 24$, *(b)* 6, 20, *(c)* 1, 120.

12.3. Decide whether H is a subgroup of G and if it is, determine the index $[G : H]$ and all (left) cosets and the transversal of H of G by H if

(a) $G = \mathbb{Z}_{12}$ and $H = \{0, 3, 6, 9\}$,

(b) $G = \mathbb{Z}_{10}$ and $H = \{0, 3, 6, 9\}$,

(c) $G = \mathbf{S}_3$ and $H = \{id, (12), (23)\}$,

(d) $G = \mathbf{S}_3$ and $H = \{id, (12)\}$.

Solutions: (a) yes, cosets: $\{0, 3, 6, 9\}$, $\{1, 4, 7, 10\}$, $\{2, 5, 8, 11\}$, *a transversal e.g.* $\{0, 1, 2\}$,
(b) no,
(c) no,
(d) yes, H, $(123)H = \{(123), (13)\}$, $(132)H = \{(132), (23)\}$, *a transversal e.g.* $\{\mathrm{id}, (123), (132)\}$.

12.4. In the group $(\mathbb{Z}, +, -, 0)$ and $a, b \in \mathbb{Z}$

(a) prove that $\langle a, b \rangle = \langle \gcd(a, b) \rangle$ for each $a, b \in \mathbb{Z}$,

(b) prove that every finitely generated subgroup of $\mathbb{Z}$ is cyclic,

(c) find a generator of the cyclic group $\langle 21, 15 \rangle$ and compute $[\mathbb{Z} : \langle 21, 15 \rangle]$,

(d) compute $[\mathbb{Z} : \langle 60, 42, 78 \rangle]$.

Solutions: (a) apply Bezout coefficients (b) use induction and (a), (c) 3, 3, *(d)* 6.

12.5. Explain, why the group $\mathbf{S}_{16}$ contains no element of order 17.

Hint: Apply the Lagrange theorem.

12.6.* Prove that the additive group of rational numbers $(\mathbb{Q}, +, -, 0)$ is infinitely generated, i.e. $\langle X \rangle_{\mathbb{Q}} \neq \mathbb{Q}$ for each finite $X \subset \mathbb{Q}$.