3 Chinese Remainder Theorem and Euler's theorem

3.1. Determine the value

(a) $\varphi(600)$,

(b) $\varphi(7425)$ (it might be useful to know that $7425 = 27 \cdot 25 \cdot 11$),

(c)* of all natural n such that $\phi(n) = 18$.

Solutions: (a) 160, *(b)* 3600, *(c)* $19, 27, 38, 54$

3.2. Calculate

(a) $3^{5^7} \bmod 28$,

(b) $100^{99^{98}} \bmod 39$,

(c)* $100^{99^{98}} \bmod 40$.

Solutions: (a) 19, *(b)* 1, *(c)* 0 *since* $40 \mid 100^2 \mid 100^{99^{98}}$.

3.3. Find the last

(a) one digits of 1357^{246},

(b) two digits of $999^{888^{777}}$ 249^{19},

(c)* three digits of 249^{19}.

Solutions: (a) 9, *(b)* 01, *(c)* 249.

3.4. Prove that $13 \mid 16^{20} + 29^{21} + 42^{22}$. (Hint: Compute modulo 13 all summands.)

3.5. Find all $x \in \mathbb{Z}$ satisfying

(a) $x \equiv 2 \pmod 3, \quad x \equiv 4 \pmod 7, \quad x \equiv 3 \pmod 8$.

(b) $2x + 1 \equiv 2 \pmod 3, \quad 3x + 2 \equiv 3 \pmod 4, \quad 4x + 3 \equiv 2 \pmod 5$.

(c) $10x \equiv 6 \pmod{32}, \quad 3x \equiv 1 \pmod 5$.

Solutions: (a) $x \equiv 2 \pmod 3 \Leftrightarrow x = 2 + 3a$ *and substitute:* $2 + 3a \equiv x \equiv 4 \pmod 7$. *Then* $a = 3 + 7b \Rightarrow x = 11 + 21b$ *for* $b \in \mathbb{Z}$, *etc. Thus* $x = 11 + 168m$ *for* $m \in \mathbb{Z}$, *(b)* $11 + 60m$ *for* $m \in \mathbb{Z}$, *(c)* $7 + 80m$.

3.6. Find all $x \in \mathbb{Z}$ such that

(a) $x^2 \equiv 1 \pmod 3, \quad x^2 \equiv 1 \pmod 7$.

(b) $x^2 \equiv -1 \pmod{66}$.

(c) $x^2 \equiv -1 \pmod{65}$.

Solutions: (a) $\{1 + 21m \mid m \in \mathbb{Z}\} \cup \{8 + 21m \mid m \in \mathbb{Z}\} \cup \{13 + 21m \mid m \in \mathbb{Z}\} \cup \{20 + 21m \mid m \in \mathbb{Z}\}$, *(b)* $\emptyset$, *(c)* $\{8+65m \mid m \in \mathbb{Z}\} \cup \{-8+65m \mid m \in \mathbb{Z}\} \cup \{18+65m \mid m \in \mathbb{Z}\} \cup \{-18+65m \mid m \in \mathbb{Z}\}$.

3.7. Using the Chinese remainder theorem, calculate 12^{100} $(mod\ 30)$.