Homework 2

Deadline: Wednesday, November 20 at 14:00.

Please submit your solutions either on paper at the beginning of the practicals or as a pdf-file in the SIS using the Study group roster (Studijní mezivýsledky) application. A maximum of 5 points can be awarded for each task. The solution to each problem must be explained. Everything that is not immediately obvious needs to be proved or quoted from lecture notes.

1. Let $\mathcal{R}$ be a domain with the quotient field $\mathcal{Q}$, $a \in R \setminus \{0, 1\}$, and $n \in \mathbb{N}$. Prove in Q that
$$\sum_{i=1}^{n} \frac{a^n}{a^i} = \frac{1-a^n}{1-a},$$
where $a^i = \prod_{j=1}^{i} a$ (the product of i copies of a).

2. Calculate $(x^4 + x^3 + x) \bmod (x^2 + 2)$ and $(x^4 + x^3 + x) \operatorname{div}(x^2 + 2)$ in $\mathbb{Z}_3[x]$.

3. Find in the field $\mathbb{Z}_5$ all roots of the polynomials
$$\text{(a) } x^5 - x, \quad \text{(b) } x^{11} + x^5 + 1, \quad \text{(c) } (x+1)^{187}.$$

4. Calculate in the domains $\mathbb{R}[x]$, $\mathbb{Q}[x]$ and $\mathbb{Z}_5[x]$ the irreducible decompositions of the polynomial $x^4 - 4$.

5. Let $\mathcal{R}[x]$ be a polynomial ring over a domain $\mathcal{R}$ and $a \in R$. Prove that $x - a$ is a prime element of $\mathcal{R}[x]$.