ON BASSIAN AND GENERALIZED BASSIAN MODULES

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Abstract. A right R-module M is called (generalized) Bassian if the existence of an injective homomorphism $M \to M/N$ for some submodule N of M implies that $N = \{0\}$ (N is a direct summand of M). We partially describe relationships between the classes of Bassian and generalized Bassian modules. In particular, we show that generalized Bassian abelian groups are precisely direct sums of Bassian and semisimple abelian groups, which is a positive answer for Conjecture 1.3 in [9].

1. Introduction

The classical theorem about modules over a Dedekind domain says that each finitely generated torsion-free module is isomorphic to a direct sum of ideals. In the case R is an integrally closed Noetherian ring, then an R-module M contains a free submodule N such that $M/N \cong I$, where I is an ideal of the ring R [2]. In [10], the authors obtained an analogue that was proved for integral rings such that each ideal has two generators, and such rings were called Bassian in their paper, which has a closed relation with the notion "Hopfian". A ring R is said to be

Hopfian if R cannot be isomorphic to a proper homomorphic image R/I ([15], [20]),

Bassian, if there cannot be an injection of R into a proper homomorphic image R/I ([19]).

As Rowen and Small pointed out in [19] that we often get more with a stronger definition, by turning to modules and abelian groups. The notion of a *Hopfian* module, which is a module M over which every epimorphism $M \to M$ is an isomorphism was studied in the pioneer paper [15]. The structural description of *Bassian* abelian groups, which are defined as groups that cannot be embedded in a proper homomorphic image of itself, and related notions was the topic of series of recent papers [5], [6], [7], [8], [9] and [16]. In this paper, following those, we say that an R-module M is

Bassian if the existence of a monomorphism $M \to M/N$, for some submodule N of M, forces that N is a trivial submodule of M,

generalized Bassian if the existence of a monomorphism $M \to M/N$, for some submodule N of M, forces that N is a direct summand of M.

2010 *Mathematics Subject Classification.* Primary 16D10, 16D40, 16D60, 20K10, 20K20, 20K21.

Key words and phrases. Abelian group, Hopfian group, (Generalized) Bassian group, (Generalized) Bassian module, Hopfian module.

Notice that the Bassian property of M is equivalent to that M cannot be embedded in a proper homomorphic image of itself, a property that was critically observed by Bass for a commutative ring in [4].

The paper consists of four sections and is organized as follows.

Section 2 is devoted to examples, counter examples and constructions which contains relationships between the classes of (generalized) Bassian modules and related module classes as (co-) Hopfian modules. For example, Bassian modules are Hopfian, and the class of Bassian modules is not closed under taking submodules and factor modules, in general.

In section 3, we consider the reverse inclusions of some examples and closure properties of the studied classes of modules.

The last section is devoted to studying relations between Bassian and generalized Bassian modules. As a consequence, we characterize generalized Bassian abelian groups as a direct sums of a Bassian and semisimple group, which gives a positive answer to [9, Conjecture 1.3]. Here, we also remark that while the article is in the refereeing process, Keef [16] introduced a partial answer for the same conjecture.

Throughout this paper we will be following the standard convention in ring theory. The rings R we consider will be associative rings with an identity element $1_R \neq 0_R$, and all right R-modules will be unitary. We write M_R to denote that M is a right R-module. We write $N \leq M$ if N is a submodule of M. A submodule N of a module M is called *proper* if $N \subsetneq M$. For a module M, we write $N \leq^{ess} M$ if N is an essential submodule of M and we shall denote the socle of M by $\mathrm{Soc}(M)$. We denote by $Sing(M)$ the singular submodule of the right (left) module M over the ring R, i.e. the subset of all elements of M whose annihilators are essential right (left) ideals of the ring R. We say that a module A is a *subfactor* of B provided there exists embedding of A into some factor of B and a homogeneous component of a module is a maximal submodule of the module such that all its simple subfactors are isomorphic to a single simple module. A module M is called *nonsingular* (*singular*) if $Sing(M) = 0$ ($Sing(M) = M$). The *torsion submodule* of M is the second singular submodule $Tor(M)$ defined by $Sing(M/Sing(M)) = Sing_2(M)$. Here a module M is called torsion-free if $Tor(M) = 0$. The category of left (resp. right) R-modules will be denoted by R-Mod (resp. Mod-R). Morphisms will be written on the side opposite to that of scalars. The $n \times n$ matrix ring over a ring R is denoted by $M_n(R)$. The symbols $\mathbb{Z}$ and $\mathbb{Q}$ denote the set of integer and rational numbers, respectively.

We refer to [1, 2, 11, 17, 18] for any undefined notion used in the text.

2. Examples

Firstly, we observe how finiteness conditions relate to different variants of the Bassian property.

Example 2.1. *Noetherian modules are Bassian. In particular, all finitely generated modules over a right Noetherian ring and every module of finite composition length are Bassian.*

Proof. We prove indirectly that each non-Bassian module is not Noetherian. Assume that M is not Bassian. Then there exist non-zero submodules, say K and N, of M such that $K \leq N$ and $N/K \cong M$. We will construct two chains of submodules

$$(K_i \mid i < \omega) \text{ and } (N_i \mid i < \omega)$$

such that $K_i \subsetneq K_{i+1}$ and $N_i \supseteq N_{i+1}$ and $N_i/K_i \cong M$, where $i < \omega$. First, we let $K_0 := K$ and $N_0 := N$. Hence if we have constructed K_i and N_i, we can find a non-zero submodule $\hat{L}$ of the factor $N_i/K_i \cong M$ for which there exists an embedding $\mu : M \to (N_i/K_i)/\hat{L}$ by the hypothesis. Now it is enough to take pre-images $K_{i+1} = \pi^{-1}(\hat{L}) \supsetneq K_i$ and $N_{i+1} = \pi^{-1}(\mu(M))$ where $\pi : N_i \to N_i/K_i$ is the natural projection. Since we have strictly increasing chain of submodules $(K_i \mid i < \omega)$ of M, we obtain that it is not Noetherian.

Finally, we note that finite length modules as well as finitely generated modules over a right Noetherian ring are Noetherian. □

The situation appears to be trivial in case of vector spaces:

Example 2.2. *Any vector space over a field is generalized Bassian, and it is Bassian if and only if it is finite-dimensional.*

It is easy to see that even arbitrary semisimple modules are generalized Bassian, and semisimple Bassian modules can be described in a simple way. First, we observe useful closure property of classes of (generalized) Bassian modules.

Lemma 2.3. *The classes of Bassian and generalized Bassian modules are closed under taking direct summands.*

Proof. The proof of the fact that a direct summand of a Bassian module is Bassian is the same as in the abelian group case [5, Proposition 2.1].

Let $M = A \oplus B$ be a decomposition of a generalized Bassian module M and let N be a submodule of A. Then there exists submodule X of M such that $M = N \oplus X$, and hence $A = (N \oplus X) \cap A = N \oplus (X \cap A)$ by the modularity. □

Example 2.4. *A semisimple module is Bassian iff all its homogeneous components are finitely generated. In particular, $\bigoplus_{p \in \mathbb{P}} \mathbb{Z}_p$ is infinitely generated Bassian abelian group, where $\mathbb{P}$ denotes the set of all prime numbers.*

Proof. Let M be a semisimple module. Assume on contrary that M contains an infinitely generated homogeneous component. Then it contains a direct summand module, say L, that is isomorphic to $S^{(\omega)}$ which is isomorphic to any its factor module L/F, where F is a finitely generated submodule. Hence $M \cong M/F$. By Lemma 2.3, the module M is not Bassian, a contradiction.

For the converse, assume that N is a submodule of M such that M embeds into M/N. Then, for each simple module S, the rank of homogeneous component of M for S is equal to the sum of corresponding ranks of M/N and N, which implies that all ranks of N are zero. Hence $N = 0$, as desired. □

A module is called *uniform* if any two of its nonzero submodules have a nonzero intersection.

A module M is called *distributive* if $A \cap (B + C) = (A \cap B) + (A \cap C)$ for all submodules A, B and C of M.

Recall that, over prime rings, distributive modules are uniform and uniform modules need not be Bassian, in general.

Example 2.5. *The Prüfer p-group $\mathbb{Z}_{p^\infty}$, for a prime number p, as a $\mathbb{Z}$-module, is a distributive artinian abelian group, which is not Bassian.*

A module M is said to be *monoform* if every non-zero homomorphism from a submodule N of M to M is a monomorphism. Notice that monoform modules are uniform by [14, Proposition 2.6 (i)].

Example 2.6. *Monoform modules are Bassian.*

Proof. Let M be a monoform module and $f : M \to M/N$ be a monomorphism. Write Im(f):= L/N for some submodules $N \leq L \leq M$. Suppose $\phi : M \to L/N$ is an isomorphism and consider the canonical projection $\eta : L \to L/N$. Clearly, $g := \phi^{-1}\eta : L \to M$ is an R-homomorphism. By the assumption, Ker(g)= 0. Hence N =Ker$(\eta) \leq$ Ker(g)= 0, as desired. □

A module is called (*co-*)*Hopfian* if any (injective) surjective endomorphism is an isomorphism [20].

Lemma 2.7. *A module M is Hopfian if and only if $N = 0$ whenever $M/N \cong M$ for some $N \leq M$.*

Proof. Assume that M is a Hopfian module. Let $\alpha : M/N \to M$ be an isomorphism for some $N \leq M$ and $\pi : M \to M/N$ be the natural epimorphism. It is easy to say that $\alpha\pi : M \to M$ is an epimorphism. Since M is Hopfian, we have $\alpha\pi$ as an isomorphism. Because α is a monomorphism, we have $Ker(\alpha\pi) = Ker(\pi)$. Then $Ker(\pi) = N = 0$.

For the converse, assume that $\alpha : M \to M$ is an epimorphism. Clearly,

$$M/Ker(\alpha) \cong M.$$

By the hypothesis, we have $Ker(\alpha) = 0$, i.e. α is an isomorphism. □

A module M is called *semi-Hopfian* if N is a direct summand of M whenever $M/N \cong M$ for some $N \leq M$ (see [3]). Note that, by Lemma 2.7, the Bassian property is a strengthening of the Hopfian property with the isomorphism being replaced by a monomorphism. The same relationship exists in the notion of generalized Bassian modules and semi-Hopfian modules.

A module M is called *Dedekind-finite* or *directly finite* if whenever N is a submodule of M such that M is isomorphic to the module $M \oplus N$, then $N = 0$. Note that the module M is Dedekind-finite if and only if for any endomorphisms f and g of M such that $fg = 1$, we have $gf = 1$.

Bassian modules are Hopfian, and generalized Bassian modules are semi-Hopfian. Furthermore, we have the following chart.

$$\begin{tikzcd}
 & \text{Generalized Bassian} \arrow[r] & \text{Semi-Hopfian} & \\
\text{Noetherian} \arrow[r] & \text{Bassian} \arrow[u] \arrow[r] & \text{Hopfian} \arrow[u] \arrow[r] & \text{Dedekind-finite}
\end{tikzcd}$$

All domains R (not necessarily commutative) are Bassian R-modules because of the Dedekind-finiteness and being monoform by [17, Corollary 8.4].

We also note that they are not Bassian as rings (for instance, [4, Example 1.2])).

Example 2.8. *Any prime ring R is Bassian as a (right) R-module.*

Proof. Let R be a prime ring and $f : R \to R/I$ be an R-monomorphism for some right ideals I of R. Write $f(1) = a + I$ for some $a \in R$. Suppose $I \neq 0$. By [17, Corollary 8.4(3)], there exist $0 \neq r \in R$ such that $ar \in I$, which implies that $f(r) = ar + I = I$ and hence $r = 0$, a contradiction. Therefore, $I = 0$. $\square$

A submodule N of a right R-module M is called *fully invariant* if $f(N)$ is contained in N for every R-endomorphism f of M.

The right R-module M is called a *duo module* provided every submodule of M is fully invariant. The ring R is called a *right duo ring* if the right R-module R is a duo module.

Note that a ring R is a right duo ring if and only if every right ideal of R is a two-sided ideal; equivalently Ra is contained in aR for every element $a \in R$. We also note that commutative rings and division rings are right (and left) duo rings.

Example 2.9. *Right (respectively, left) duo rings are Bassian as right (respectively, left) modules.*

Proof. Let R be a right duo ring, and $f : R \to R/I$ be an R-monomorphism for some right ideals I of R. Suppose $I \neq 0$. Then there exist $0 \neq x \in I$. Now f is a right R-linear map, then we have $f(1) = a + I$ for some $a + I \in R/I$. Hence $f(x) = f(1)x = (a + I)x = ax + I = I$ since I is a two-sided ideal. But f is a monomorphism, which gives $x = 0$, a clear contradiction. Therefore, $I = 0$, as desired. $\square$

Note that duo modules need not be Hopfian or co-Hopfian, in general. Moreover, Hopfian modules need not be duo and neither need co-Hopfian modules be duo.

The following two examples show that the class of Bassian modules is not closed under taking submodules and factor modules, in general.

Example 2.10. $\mathbb{Q}_{\mathbb{Z}}$ *is a Bassian abelian group and any its non-trivial factor group is not Bassian as it contains a direct summand isomorphic to non-Bassian group* $\mathbb{Z}_{p^\infty}$ *for some prime number p.*

Example 2.11. *Let p be any prime integer and let S be the trivial extension of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ by $\mathbb{Z}$, i.e. $S = \mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$. Firstly, we note that S is a commutative ring and hence S is a Bassian S-module. On the other hand, S has a submodule which is not Bassian.*

Proof. Take $I := \{(0, m) : m \in \mathbb{Z}_{p^\infty}\}$. It is easy to see that I is an ideal of S and $\theta : I \to I$, defined by $\theta(0, m) = (0, pm)$, is an S-linear map. Moreover, θ is surjective, but it is not injective (because if we take $m = \frac{1}{p} + \mathbb{Z} \in \mathbb{Z}_{p^\infty}$, then $\theta(0, m) = (0, pm) = (0, 0) \in I$. Hence we have $I/Ker(\theta) \cong I$ as an S-linear map). This shows that I is not a Bassian S-module. $\square$

A collection of examples where submodules of a Bassian module are not Bassian can be given as follows.

Example 2.12. *Let R be a commutative ring and M be an R-module which is not Bassian. Then there exists a surjection $\phi : M \to M$ which is not an isomorphism. Now consider S be the trivial extension ring of M by R i.e., $S = R \oplus M$. Firstly, we note that S is a commutative ring and hence S is a Bassian S-module. Write $I = \{(0, m) : m \in M\}$. Then I is an ideal of S. Now define the map $f : I \to I$ by $f(0, m) = (0, \phi(m))$. Clearly, f is an S-linear map which is surjective but not injective. Hence $I/Ker(f) \cong I$. This shows that I is not a Bassian S-module.*

A submodule N of a module M is said to be *dense* if, for any $y \in M$ and $x \in M\backslash\{0\}$, there exists $r \in R$ such that $xr \neq 0$, and $yr \in N$ ([17, Definition 8.2]).

Example 2.13. *Modules whose all proper submodules are dense are Bassian.*

Proof. Assume that M is an R-module such that all its proper submodules are dense submodules. If N a dense submodule of M, then, for any submodule P such that $N \leq P \leq M$, we have $Hom_R(P/N, M) = 0$ by [17, Proposition 8.6]. □

A concrete non-trivial (i.e. non-semisimple) example of a Bassian module where all submodules are Bassian can be seen in the following by precisely highlighting a pair of non isomorphic submodules.

A module M is *critical* provided M has Krull dimension and all proper quotient modules of M have Krull dimension strictly less than the Krull dimension of M ([14, §2]).

Note that, every critical module is monoform (see [14, Corollary 2.5]).

Example 2.14. ([13, p.1846-1848]) *We consider the formal power series ring $S = K[[t]]$ an indeterminate t, where K a field of characteristic zero. Let $R = S[\theta]$ be the formal linear differential operator ring (i.e. the Ore extension) over (S,δ), where δ is a K-linear derivation on S. Since*

$$s\theta = \theta s - \delta(s)$$

for all $s \in S$, the ring S can be made into a right R-module isomorphic to $R/\theta R$ with a module multiplication $*$ defined by

$$s * s' = ss' \text{ and } s * \theta = -\delta(s)$$

for all $s, s' \in S$. Note that

$$t^n * \theta = -nt^n$$

for all $n \in \mathbb{N}$, which gives that the ideals $t^n S$ are also right R-submodules (pairwise non-isomorphic) of S. Thus, the nonzero right R-submodules of S form a strictly descending chain

$$S > tS > t^2 S > \cdots,$$

which implies that S (and hence, $R/\theta R$) is a 1-critical right R-module of Krull dimension 1 (see [14, §2]). Now,

(a) S is a monoform module and hence a Bassain right R-module by Example 2.6, and

(b) each $t^i S$, $i \in \mathbb{N}$, is a critical R-submodule of S of Krull dimension 1 (see [14, Proposition 2.3]), which gives that all of these submodules are Bassian.

3. Classes of Bassian modules

We begin with a partial description for (generalized) Bassian modules. Note that, the equivalence between (1) and (2) for the case "the generalized Bassian" in the following lemma is similar.

Lemma 3.1. *The following statements are equivalent for a module M.*

(1) M is Bassian (respectively, generalized Bassian).

(2) For any R-module M', if there exixts a monomorphism $\sigma : M \to M'$, then the kernel of any epimorphism $\pi : M \to M'$ is 0, i.e. π is an isomorphism (respectively, $\pi : M \to M'$ is a direct summand of M).

Proof. (1) $\Rightarrow$ (2). Assume that there exists a monomorphism $\sigma : M \to M'$ and an epimorphism $\pi : M \to M'$. Clearly, there is an isomorphism $\alpha : M' \to M/Ker(\pi)$, and hence $\alpha\sigma : M \to M/Ker(\pi)$ is a monomorphism. Since M is a Bassian module, we obtain that $Ker(\pi) = 0$, i.e. π is an isomorphism.
(2) $\Rightarrow$ (1). Let $f : M \to M/N$ be a monomorphism for some submodule N of M and consider the natural epimorphism $\pi : M \to M/N$ be. By the hypothesis, we have N =Ker(π)= 0, as desired. □

Proposition 3.2. *The following statements are equivalent for a module M.*

(1) M is Bassian;

(2) M is generalized Bassian and there exists no a monomorphism from M to any proper direct summands of M.

Proof. (1) $\Rightarrow$ (2). Clearly, M is generalized Bassian. Assume on contrary that $f : M \to N$ is a monomorphism, where N is a proper direct summand of M. Write $M = N \oplus N'$ for some $N' \leq M$. Clearly, $gf : M \to M/N'$ is a monomorphism, where $g : N \to M/N'$ is the natural isomorphism. Since M is Bassian, we get $N' = 0$ which shows that $M = N$, a contradiction.
(2) $\Rightarrow$ (1). Let $\sigma : M \to M/N$ be a monomorphism for some submodule N of M. Since M is generalized Bassian, we obtain that N is a direct summand of M. Write $M := N \oplus N'$ for some $N' \leq M$. Then there exist a monomorphism $\theta \circ \sigma : M \to N'$, where $\theta : M/N \to N'$ is the natural isomorphism. Therefore $N' = M$ by (2), and hence $N = 0$. □

It has already been noted that there exist uniform modules which are not Bassian. On the other hand, by Example 2.6, monoform modules are Bassian.

Proposition 3.3. *Nonsingular uniform modules are monoform. Hence, nonsingular uniform modules are Bassian.*

Proof. Let M be a nonsingular and uniform module. We prove that M is monoform. Consider a submodule N of M and $f : N \to M$. If $f = 0$, then the module M is monoform. Suppose $f \neq 0$, i.e. Im(f)$\neq$ 0. In case, Ker(f)$\neq$ 0, we have Ker(f)$\leq^{ess}$ N since M is uniform. Hence N/Ker(f)$\cong$Im(f) which implies that Im(f) is a singular module. This is a contradiction. Therefore, Ker(f)= 0, i.e. M is a monoform module.

Although the second part directly follows from Example 2.6, we give a direct proof for the claim: Again, let M be a nonsingular and uniform module. Suppose on contrary that M is not Bassain. By Lemma 3.1, there exist a triple (M', f, θ), where M' is a module, $f : M \to M'$ is a monomorphism and $\theta : M \to M'$ is an epimorphism with Ker(θ)$\neq 0$. Call $K :=$Ker(θ). Since M is uniform, we have $K \leq^{ess} M$. Hence $M/K \cong M'$ which implies that M' is a singular module, i.e. Im(f) is a singular submodule of M'. Since Im(f)$\cong M$ (via ϕ say), we obtain that $\phi(Sing(\text{Im}(f)) = \phi(\text{Im}(f) \subseteq Sing(M) = 0$, a contradiction. □

A module M is said to be *polyform* if Ker(f) is a closed submodule of K, for every submodule K of M and $f : K \to M$ [11]. Notice that any nonsingular module is polyform.

Proposition 3.4. *Both uniform and polyform modules are Bassian.*

Proof. Assume that M is a polyform and uniform module. Let $f : M \to M/N$ be a monomorphism. Write Im(f):$= L/N$ for some submodules $N \leq L \leq M$. Let $\phi : M \to L/N$ be the isomorphism and consider the canonical projection $\eta : L \to L/N$. Clearly, $g = \phi^{-1}\eta : L \to M$ is an R-homomorphism. By the assumption, Ker(g) is a closed submodule of L. But L is uniform, we have either Ker(g)$= L$ or Ker(g)$= 0$ by [17, Examples 6.42(1)]. If Ker(g)$= L$, we get contradiction. Therefore Ker(g)$= 0$ which implies that η is a monomorphism. Hence $N = 0$, as required. □

Proposition 3.5. *The following statements are equivalent for a module M.*

(1) M is Hopfian.
(2) M is semi-Hopfian and Dedekind-finite.

Proof. (1) $\Rightarrow$ (2). The implication is clear.
(2) $\Rightarrow$ (1). Assume that there exists an isomorphism between M and M/N for some $N \leq M$. Since M is semi-Hopfian, we obtain that N is a direct summand of M. Write $M := N \oplus N'$, where $N' \leq M$. Now we have

$$M = N \oplus N' \cong N \oplus M/N \cong N \oplus M.$$

The Dedekind-finiteness of the module M gives that $N = 0$, as desired. □

The following assertion collects some properties of (generalized) Bassian modules.

Proposition 3.6. *Let M be a module.*

(1) If M is free and Bassian, then it is of finite rank.
(2) If M has a submodule, say N, such that $N/X \cong M$ for some $0 \neq X \leq N$, then M is not Bassian.
(3) If M/N is Bassian for every nonzero submodule N of M, then M is Hopfian.

Proof. (1) The claim follows from [15, Proposition 12].
(2) If M has a submodule N such that $N/X \cong M$ for some $0 \neq X \leq N$, then certainly M embeds in M/X, and hence M cannot be Bassian because $X \neq 0$.
(3) Assume on contrary that M is not Hopfian. Then there exists a surjection, say f, of M which is not an isomorphism. Let $N :=$ Ker$f$. Clearly, $0 \neq N$ and f induces an isomorphism, say $\bar{f} : M/N \to M$. Consider the canonical quotient morphism $\eta : M \to M/N$. It is easy to check that $\eta\bar{f} : M/N \to M/N$ is an epimorphism which is not an isomorphism, a contradiction. □

Recall that two modules are *Morita equivalent* if their module categories are equivalent as exact categories.

Theorem 3.7. *Being Bassian for modules is Morita invariant.*

Proof. Let R and S be Morita equivalent rings with inverse category equivalences $\alpha : Mod\text{-}R \to Mod\text{-}S$ and $\beta : Mod\text{-}S \to Mod\text{-}R$. Suppose $M \in Mod\text{-}R$ is Bassian. We must show that $\alpha(M)$ is Bassian. Take $\alpha(M') \in Mod\text{-}S$. Let $f : \alpha(M) \to \alpha(M')$ be an S-module monomorphism and $\theta : \alpha(M) \to \alpha(M')$ be an S-module epimorphism. Since any category equivalence preserves monomorphisms and epimorphisms, we obtain $\beta(f) : \beta\alpha(M) \to \beta\alpha(M')$ is an R-module monomorphism and $\beta(\theta) : \beta\alpha(M) \to \beta\alpha(M')$ is an R-module epimorphism. Note that $\beta\alpha(M) \cong M$ and $\beta\alpha(M') \cong M'$. Hence there exist a monomorphism $\bar{f} : M \to M'$ and an epimorphism $\bar{\theta} : M \to M'$. By Lemma 3.1, we have Ker$(\bar{\theta})$= 0, i.e. Ker(θ)=0, as required. □

Corollary 3.8. *Let $n \geq 2$. The following statements are equivalent for a ring R.*

(1) *Every n-generated R-module is Bassian.*
(2) *Every cyclic $M_n(R)$-module is Bassian.*

Proof. Let $P = (R^n)_R$ and $S = End(P_R)$. Then

$$Hom_R(P, -) : N_R \mapsto Hom_R({}_SP_R, N_R)$$

defines a Morita equivalence between $Mod\text{-}R$ and $Mod\text{-}S$ with the inverse equivalence

$$- \otimes_S P : X_S \mapsto X \otimes P.$$

Note that, $Hom_R(P, N)$ is a cyclic S-module for any n-generated R-module N, and $M \otimes_S P$ is an n-generated R-module for any cyclic S-module M. Thus, every cyclic S-module is Bassian if and only if every n-generated R-module is Bassian by Proposition 3.7. □

Recall that if $\phi : R \to S$ is a ring homomorphism (with $\phi(1) = 1$) and M is a right S-module, then M_ϕ denotes the right R-module M by pull back along ϕ, i.e. $mr = m\phi(r)$ for $m \in M$ and $r \in R$ (see [12, p. 334]).

Proposition 3.9. *Suppose that $\phi : R \to S$ is a ring homomorphism and M is an S-module.*

(1) *If M_ϕ is Bassian, then so is M_S.*
(2) *If ϕ is surjective, then M_S is Bassian iff M_ϕ is Bassian.*

Proof. (1) Assume that $f : M \to M/N$ is a monomorphism in $Mod\text{-}S$. Then $f : M_\phi \to (M/N)_\phi$ is a monomorphism in $Mod\text{-}R$. Hence, by the hypothesis, $N_\phi = 0$, i.e. $N = 0$, as desired.
(2) Assume that $f : M_\phi \to M_\phi/N_\phi$ is an R-monomorphism. Since ϕ is surjective and $f(ms) = f(m\phi(r)) = f(mr) = f(m)r = f(m)\phi(r)$, we see that f is an S-module monomorphism. Hence $N = 0$, i.e. $N_\phi = 0$. □

Corollary 3.10. *Let I be an ideal of a ring R and M be a right R-module. If M/MI is Bassian as an R/I-module, then M is Bassian as a right R-module.*

A ring R is said to be *right Goldie* if $u.dim(R_R) < \infty$ and R has ACC on right annihilator ideals.

Proposition 3.11. *Let R be a semiprime right Goldie ring, Q be a classical right ring of fractions of R and M be a right R-module. Then the (right) Q-module $M \otimes_R Q$ with finite uniform dimension is Bassian.*

Proof. By [17, Theorem 11.13], Q is semisimple. Hence, the reduced rank $\rho(M)$ is defined as $\rho(M) = u.dim(M \otimes_R Q)$ by [17, Proposition 11.15]. Moreover, for any submodule N of M, we have $\rho(M) = \rho(N) + \rho(M/N)$ by [17, Theorem 7.38]. Now if there exists a Q-monomophism $f : M \otimes_R Q \to (M \otimes_R Q)/(N \otimes_R Q)$, then $u.dim(N \otimes_R Q) = 0$, i.e. $N \otimes_R Q = 0$. Hence $M \otimes_R Q$ is a Bassian Q-module. □

Let M be a right R-module. The elements of the module $M[X]$ with a commuting indeterminate X over R are formal sums of the form

$$\sum_{i=0}^{k} m_i X^i = m_0 + m_1 X + m_2 X^2 + \cdots + m_k X^k$$

with $k \geq 0$ an integer and $m_i \in M$. Addition is defined by adding the corresponding coefficients. The $R[X]$-module structure is given by

$$(\sum_{j=0}^{k} m_j X^j).(\sum_{i=0}^{l} r_i X^i) = \sum_{t=0}^{k+l} \bar{m}_t X^t,$$

where $\bar{m}_t = \sum_{j+i=t} m_j r_i$ for any $m_j \in M$ and $r_i \in R$.

Proposition 3.12. *Let $M \in$ Mod-R. If $M[X]$ is Bassian in Mod-$R[X]$, then M is Bassian in Mod-R.*

Proof. Let $f : M \to M'$ be a monomorphism in Mod-R and $g : M \to M'$ be an epimorphism in Mod-R. Then one can check that

$$f[X] : M[X] \to M'[X] \text{ given by } \; f[X](\sum_{i=0}^{k} m_i X^i) = \sum_{i=0}^{k} f(m_i) X^i$$

is a monomorphism in Mod-$R[X]$ and

$$g[X] : M[X] \to M'[X] \text{ given by } \; g[X](\sum_{i=0}^{n} m_i X^i) = \sum_{i=0}^{n} g(m_i) X^i$$

is an epimorphism in Mod-$R[X]$. Since $M[X]$ is Bassian in Mod-$R[X]$, we obtain that Ker($g[X]$)= 0, which implies Ker(g)= 0. □

Let us remark that a module M over a domain R is divisible iff $M = Mr$ for any $0 \neq r \in R$, and a module M over a domain R is torsion-free iff $mr \neq 0$ for any $0 \neq m \in M$ and $0 \neq r \in R$.

Proposition 3.13. *Let R be a commutative domain and M be a divisible R-module. If M is Bassian, then M is torsion-free.*

Proof. Let $t \in \mathrm{Tor}(M)$. Then there exist $0 \neq r \in R$ such that $rt = 0$. Now, for the fixed $r \in R$, define $\theta : M \to M$ by $\theta(m) = rm$ for all $m \in M$. Then θ is an R-linear map. Moreover, for each $n \in M$, there exists $n' \in M$ such that $rn' = n$ (since M is a divisible module), which gives that θ is onto. Since M is a Bassian module, we have θ is a monomorphism. Therefore, $\theta(t) = rt = 0$, i.e. $t = 0$. Thus $\mathrm{Tor}(M) = 0$, as desired. □

Remark 3.14. *Torsion-free Bassian R-modules M need not be divisible, in general (take for example $R = M = \mathbb{Z}$).*

4. Bassian vs. generalized Bassian modules

The main theme of this section can be described as a study of the relations between classes of Bassian modules and generalized Bassian modules.

We start with one particular case when the classes coincide.

Proposition 4.1. *The following statements are equivalent for a co-Hopfian module M.*

1. *(1) M is generalized Bassian.*
2. *(2) M is Bassian.*

Proof. (2) $\Rightarrow$ (1). The implication is clear.
(1) $\Rightarrow$ (2). Let $N \leq M$ and $\sigma : M \to M/N$ be a monomorphism. Since M is generalized Bassian, we obtain that N is a direct summand of M. Let $M = N \oplus N'$ for some $N' \leq M$. Clearly, there exists an isomorphism between M/N and N', say α, and $\alpha\sigma : M \to N'$ is a monomorphism. Since M is co-Hopfian, we get that $\alpha\sigma$ is an isomorphism. Since α is a monomorphism, we have $Ker(\alpha\sigma) = Ker(\sigma)$. So,

$$Ker(\alpha\sigma) = Ker(\sigma) = N = 0$$

because σ is a monomorphism. Hence M is Bassian. □

The following proposition offers a way how to construct new examples of Bassian modules.

Proposition 4.2. *Let F_i, $i \in I$, be finitely generated modules with the essential socle over a Noetherian ring R such that*

$$Hom_R(Soc(F_i), F_j/K) = 0$$

for each submodule K of F_j with $i \neq j$. Then $\bigoplus_{i\in I} F_i$ is a Bassian module.

Proof. Let $M := \bigoplus_{i\in I} F_i$. Note that $\mathrm{Soc}(M) = \bigoplus_{i\in I} \mathrm{Soc}(F_i)$ and it is essential in M. Let N be a submodule of M and $\nu : M \to M/N$ be a monomorphism. If there exists i such that $\nu(F_i) \nsubseteq F_i + N/N$, then there exist $j \neq i$ and nonzero homomorphisms

$$\mathrm{Soc}(F_i) \to F_j + N/N \cong F_j/(N \cap F_j),$$

a contradiction. Hence

$$\nu(F_i) \subseteq F_i + N/N \cong F_i/(N \cap F_i)$$

and

$$N \cap F_i = 0$$

by Example 2.1. They imply that

$$N \cap \mathrm{Soc}(M) = \bigoplus_{i \in I} N \cap \mathrm{Soc}(F_i) = 0.$$

Thus $N = 0$, i.e. M is Bassian. □

Corollary 4.3. *A torsion module over a Dedekind domain with finitely generated homogeneous components is Bassian.*

Proof. Since any torsion module over a Dedekind domain decomposes into a direct sum of homogeneous components, the assertion follows from Proposition 4.2. □

The next technical observation allows us to say more about the structure of generalized Bassian modules.

Proposition 4.4. *Let M be a generalized Bassian module and $\mu : M \to M/K$ be an embedding where $K \leq M$. Then*

1. *K is semisimple,*
2. *$K^{(\omega)}$ is embeddable into M,*
3. *there exist submodules S and N such that $K \leq S$ is semisimple, $\mu(M)$ is an essential submodule of N and $M = S \oplus N$.*

Proof. (1) It is enough to show that K is completely reducible. Let $A \leq K$. Then there exists a decomposition $M = N \oplus K$ such that M is embeddable into $N \cong M/K$ because M is generalized Bassian. Thus M is also embeddable into $M/A \cong N \oplus K/A$, which implies that there exists a direct summand B of M such that $M = A \oplus B$. Now, by modularity, we get

$$K = M \cap K = (A \oplus B) \cap K = A \oplus (B \cap K),$$

which proves that K is completely reducible and so semisimple.
(2) Using (1), we can construct a pair of chains

$$(K_i \mid i < \omega) \text{ and } (N_i \mid i < \omega)$$

of submodules such that $K \cong K_{i+1} \leq N_i$ is a direct summand of N_i, M is embeddable into N_i, $N_i \supseteq N_{i+1}$ and $N_i \cap \sum_{j \leq i} K_j = 0$ for each i. Let $K_0 := K$. Assume that $N_0 \cong M/K$ is a direct complement of K in M, which contains a submodule, isomorphic to M, by the hypothesis. Suppose K_i and N_i are defined. Since there exist an embedding $\nu : M \to N_i$, a submodule K_{i+1}, isomorphic to K, is a direct summand in $\nu(M)$, we have a submodule N_{i+1} satisfying $\nu(M) = K_{i+1} \oplus N_{i+1}$. Now it is easy to see that $K^{(\omega)} \cong \bigoplus_{i<\omega} K_i$ is embeddable into M.
(3) Let S be a maximal submodule of M containing K such that $S/K \cap \mu(M) = 0$. Then the composition $\pi\mu$, where $\pi : M/K \to M/S$ is the natural projection, forms an embedding. Thus S is a semisimple direct summand of M containing K, and hence there exists N for which $\mu(M) \leq N$. Now, by the maximality of the choice of S, we get that $\mu(M)$ is essential submodule of N. □

Now we can formulate an analogue of Proposition 4.1.

Proposition 4.5. *Let M be a module such that every homogeneous component of $Soc(M)$ is finitely generated. The following statements are equivalent.*

(1) *M is generalized Bassian.*
(2) *M is Bassian.*

Proof. If M is generalized Bassian and, for a submodule K, there exists embedding $\mu : M \to M/K$, then by Lemma 4.4 K is semisimple and every nonzero for simple submodule S of K, there exists embedding $s^{(\omega)}$ into M. As every homogeneous component of $\mathrm{Soc}(M)$ is finitely generated, $K = 0$, and so M is Bassian. □

Corollary 4.6. *The following statements are equivalent for a module M with the zero socle.*

(1) *M is generalized Bassian.*
(2) *M is Bassian.*

The following series of observations describe generalized Bassian modules which are the sum of a Bassian module and a semisimple module. Of course, we recall that exactly semisimple modules with finitely generated homogeneous components are Bassian by Example 2.4.

Lemma 4.7. *Let $\pi : M \to H$ be a homomorphism of modules where H is semisimple.*

(1) *If $K \subseteq Soc(M)$ and $K \cap Ker(\pi) = 0$, then $K \cong \pi(K)$ and there exists a decomposition $M = A \oplus K$.*
(2) *There exist decompositions $M = A \oplus D$ and $H = \pi(D) \oplus Y$ such that $\pi(D) = \pi(Soc(M))$, $\pi(Soc(A)) = 0$ and $Ker(\pi) \cap D = 0$.*

Proof. (1) Since $\pi(K)$ is a direct summand of the semisimple module H, there exist an epimorphism $\rho : M \to \pi(K)$ (induced by π) and a monomorphism $\nu : \pi(K) \to M$ such that $\nu(\pi(K)) = K$, $\pi\nu = \mathrm{id}_{\pi(K)}$ and $\mathrm{Ker}(\pi) \subseteq \mathrm{Ker}(\rho)$. Now it is easy to see that the module M has a decomposition $M = A \oplus K$, where $A := \mathrm{Ker}(\rho)$.
(2) As $\pi(\mathrm{Soc}(M))$ is isomorphic to a direct summand, say D, of $\mathrm{Soc}(M)$, it is enough to apply (1) on $K = D$. □

We say that a module A is *directly bounded* if there exists a finite n such that, for each simple submodule T of A, there exist no a direct summand of A which is isomorphic to $T^{(n)}$.

Lemma 4.8. *Let B, H and F be submodules of a module M such that B contains no a direct simple summand, H is semisimple and $M/F \cong B \oplus H$.*

If there exists a finite n such that for each simple module T which has no a factor of F is isomorphic to $T^{(n)}$, then there exists a decomposition $M = A \oplus D$ such that A is directly bounded and D is semisimple.

Proof. Denote by

$$\pi : M \to M/F \cong B \oplus H$$

ensured by the hypothesis and $\pi_B : M \to B$, $\pi_H : M \to H$ the compositions of π and the corresponding natural projections. Then, by Lemma 4.7, there exist decompositions $M = A \oplus D$ and $H = \pi(D) \oplus Y$ such that $\pi(\mathrm{Soc}(A)) \subseteq B$ and π acts injectively on D. Thus the factorization $\pi(D)$ gives us an epimorphism $\rho : A \to B \oplus Y$ with $\mathrm{Ker}(\rho) = F$ and $\rho(\mathrm{Soc}(A)) \subseteq B$.

It remains to prove that A is directly bounded. Assume that, there exists a simple module, say T, and submodules $L \subseteq \mathrm{Soc}(A)$ and $X \subseteq A$ such that $L \cong T^{(n)}$ and $A = L \oplus X$. Then

$$\rho(A) = \rho(X) + \rho(L),$$

where $\rho(L)$ is semisimple, which implies $\rho(A) = \rho(X) \oplus U$ for suitable submodules $U \subseteq \rho(L) \subseteq \rho(\mathrm{Soc}(A)) \subseteq B$. Thus

$$B = (U \oplus \rho(X)) \cap B = U \oplus (\rho(X) \cap B)$$

by the modularity. Since B contains no a simple direct summand, we obtain that $U = 0$ and so $\rho(A) = \rho(X)$. Now, this implies that $A = F + X = L \oplus X$ and $T^{(n)} \cong L \cong A/X \cong F/(F \cap M)$, which contradicts to the hypothesis that no factor of F is isomorphic $T^{(n)}$. □

Lemma 4.9. *Assume M and S are modules such that S is semisimple, N is a submodule of $M \oplus S$ and $\nu : M \to (M \oplus S)/N$ is an embedding.*

Then there exist submodules, say T and L, of the module S and a decomposition $M = A \oplus D$ such that $M \oplus S = (M + N) \oplus T \oplus L$, M_1 embeds into $M/(M \cap N)$, and $D \cong T$.

Proof. Since $(M + N) \cap S$ is a submodule of a semisimple module S, we can find a direct summand, say U, such that $S = U \oplus ((M+N) \cap S)$. By the modularity we get that $(U+(M+N)) \cap S = U + ((M+N) \cap S) = S$, which implies $M+S \subseteq U+M+N$. Thus $M \oplus S = (M+N) \oplus U$ and so $(M \oplus S)/N = (M+N) \oplus U \cong U \oplus M/(M \cap N)$. Let us denote $\pi_1 : (M \oplus S)/N \to U$ and $\pi_2 : (M \oplus S)/N \to M/(M \cap N)$ the natural projections, $\nu_i = \pi_i \nu$ and $K_i = \mathrm{Ker}\nu_i$ for $i = 1, 2$. Note that $K_1 \cap K_2 = 0$ as ν is injective, hence there exists submodules $D = K_2$ and A of M satisfying $M = A \oplus D$ and $D \cong \nu_1(D)$ by Lemma 4.7.

As $A \cap K_2 = 0$, the homomorphisms ν_2 embeds A into $M/(M \cap N)$. Similarly, since $S_1 \cap K_1 \subseteq K_2 \cap K_1 = 0$, the homomorphisms ν_1 embeds D into the semisimple module U. Hence, there exists a decomposition $U = T \oplus L$ such that $D \cong T$. □

Now we can formulate an assertion employing all technical results of this section.

Theorem 4.10. *Let M be a directly bounded Bassian module and S a semisimple module. Then $M \oplus S$ is generalized Bassian.*

Proof. Let N be a submodule of $M \oplus S$ and let $\nu : (M \oplus S) \to M \oplus S/N$. First, we define $M_0 = M$, $S_0 = S$, $N_0 = N$ and let ν_0 denote the restriction $\nu \mid_M$ of the monomorphism ν. Then Lemma 4.9 allows us to construct, by the induction on $n > 0$, sequences of modules M_i, S_i, T_i, L_i, N_i such that

$$M_{i-1} = M_i \oplus S_i, \quad N_i = N_{i-1} \cap M_{i-1}, \quad M_i \oplus S_i = (M_i + N_i) \oplus T_i \oplus L_i$$

satisfying that $L_{i-1} \cong S_i$ M_i embeds into M_{i-1}/N_i and S_i embeds into S_{i-1}.

Since M is directly bounded, we can fix n for which there is no a direct summand of M which is isomorphic to $T^{(n)}$ for an arbitrary simple module, say T. We will show that the sequence of direct summands S_i terminates in the n-th step by the zero module. To prove it, let us assume on contrary that $S_{n-1} \neq 0$ and T is a simple submodule of S_{n-1}. Since S_i is embeddable into S_{i-1} for each i, we get that an isomorphic copy of T is a submodule of S_i for each $i < n$. Hence $T^{(n)}$ is isomorphic to a direct summand of the semisimple module $\bigoplus_{i<n} S_i$, which

is a contradiction because $M = M_0 = M_{n-1} \oplus \bigoplus_{i<n} S_i$. Thus $S_{n-1} = 0$ and $M_{i-2} = M_{i-1} \oplus S_{n-1} = M_{i-1}$.

Note that M_0 is Bassian by the hypothesis and M_i is a direct summand of the Bassian module M, and hence it is Bassian by Lemma 2.3 for each $i > 0$. Since the Bassian module M_{n-1} is embeddable into $M_{n-2}/N_{n-1} = M_{n-1}/N_{n-1}$, we get that $N_{n-1} = 0$. Thus, we have k such that $S_{k+1} = 0$, $N_k = 0$, and $M = M_k \oplus \bigoplus_{i\leq k} S_i$. Since $M_k = \bigcap_{i=0}^{k} M_i$ and $N_i = N_{i-1} \cap M_i$, we get that

$$N_0 \cap M_k = N_0 \cap \bigcap_{i=0}^{k} M_i = (N_0 \cap M_0) \cap \bigcap_{i=1}^{k} M_i = N_1 \cap \bigcap_{i=2}^{k} M_i = \cdots = N_k = 0$$

and $\prod M \to \bigoplus_{i\leq k} S_i$ is a projection such that $\mathrm{Ker}(\pi) = M_k$ and $N_0 \cap M_k = 0$. Now, Lemma 4.7 implies that $N = N_0$ is a direct summand of $M \oplus S$. □

Corollary 4.11. *If M is a Bassian module and S is a semisimple module such that there exists n such that either all dimensions of homogeneous components either of socle of M or of S are directly bounded by n, then $M \oplus S$ is generalized Bassian.*

The following lemma may be well known, but we give a proof for completeness.

Lemma 4.12. *Let M_i, $i \in I$ be modules with a finitely generated socle. Then, there exists a decomposition $\bigoplus_{i\in I} M_i = B \oplus H$ such that B contains no a simple direct summand and H is semisimple.*

Proof. Since $\mathrm{Soc}(M_i)$ is finitely generated, there exists a maximal semisimple direct summand of M_i, i.e. there is a decomposition $M_i = B_i \oplus H_i$ where B_i contains no a simple direct summand and H_i is semisimple, $i \in I$. Put $B := \bigoplus_{i\in I} B_i$ and $H := \bigoplus_{i\in I} H_i$. Now, it is easy to see that $\bigoplus_{i\in I} M_i = B \oplus H$, where H is semisimple. In order to prove that B contains no a simple direct summand, we assume that C is a maximal submodule of B, which is a direct summand if B. Hence, there exist $i \in I$ and a simple submodule $S \subseteq B_i$ for which $B = S \oplus C$. Now, by the modularity, we obtain that $B_i = (S \oplus C) \cap B_i = S \oplus (C \cap B_i)$, which is a contradiction. □

The following assertion is the main result of this section.

Theorem 4.13. *Assume that M is a Bassian module, F is a submodule of M and M_i, $i \in I$, are modules with finitely generated socles. If there exists a finite n such that, for each simple module T, no a factor of F is isomorphic to $T^{(n)}$, then*

(1) there exist a decomposition $M = A \oplus D$ such that A is a directly bounded Bassian module and H is a semisimple module with finitely generated homogeneous components,
(2) $M \oplus S$ is a generalized Bassian module for every semisimple module S.

Proof. (1) The claim follows from Lemma 4.12 that M/F decomposes into a direct sum of a submodule containing no a simple direct summand and a semisimple module. Then the assertion is implied by Lemma 4.8.
(2) Since $M = A \oplus D$ for a directly bounded Bassian module A and a semisimple module D by (1), we obtain that the module $M \oplus S \cong A \oplus (D \oplus S)$ is generalized Bassian by Theorem 4.10. □

The following consequence answers the Conjecture 1.3 in [9] in positive.

Corollary 4.14. *If A is a Bassian abelian group and S is a semisimple abelian group, then $A \oplus S$ is generalized Bassian.*

Proof. By [5, Main Theorem], there exists a free subgroup, say F, of finite rank $n-1$ of A such that $\mathrm{Tor}(A) \oplus F$ is essential in A. Since all ranks of the group A are finite and A/F is a torsion group, we obtain that all p-components has finite socles and no a factor of F is a homogeneous semisimple module of rank n. Hence the claim follows from Theorem 4.13(2). □

Finally, we formulate the characterization of the generalized Bassian property summing up results of the paper [9] and of the present paper.

Corollary 4.15. *The following statements are equivalent for an abelian group A.*

(1) *A is generalized Bassian,*
(2) *There exists a decomposition $A = B \oplus H$ such that A is Bassian and H is elementary,*
(3) *There exists decomposition $A = B \oplus H$ such that A is directly bounded Bassian and H is elementary.*

Proof. (1)⇒(2) The implication is proved in [9, Corollary 3.6].
(2)⇒(3) The implication follows from Theorem 4.13(1).
(3)⇒(1) The implication is an immediate consequence of Corollary 4.14. □

Acknowledgments We sincerely thank the reviewer for their thoughtful, encouraging, and detailed feedback, which we found both valuable and motivating. The first-named author is thankful to IISER Berhampur for providing facilities to carry out this work with a Postdoctoral Fellowship.

References

[1] F.W. Anderson, K.R. Fuller, Rings and Categories of Modules. Second edition Grad. Texts in Math., 13 Springer-Verlag, New York, 1992.
[2] N. Bourbaki, Commutative Algebra, Addison-Wesley, 1972.
[3] P. Aydogdu, A.Ç. Özcan, Semi co-Hopfian and semi Hopfian modules, East West J. Math. 10(1)(2008), 57–72.
[4] H. Bass, A finiteness property of affine algebras, Proc. Amer. Math. Soc. 110(2)(1990), 315–318.
[5] A.R. Chekhlov, P.V. Danchev, B. Goldsmith, On the Bassian property for abelian groups, Arch. Math. (Basel) 117(6)(2021), 593–600.
[6] A.R. Chekhlov, P.V. Danchev, B. Goldsmith, On the generalized Bassian property for abelian groups, Acta Math. Hungar. 168(1)(2022), 186–201.
[7] A.R. Chekhlov, P.V. Danchev, P.W. Keef, Semi-generalized Bassian groups, J. Group Theory, 28 (2025), 391–407.
[8] A.R. Chekhlov, P.V. Danchev, P.W. Keef, Generalizations of the Bassian and co-Bassian Properties for abelian groups, J. Commut. Algebra 17(1)2025, 1-11.
[9] P.V. Danchev, P.W. Keef, Generalized Bassian and other mixed abelian groups with bounded p-torsion, J. Algebra 663(1)2025, 1-19.
[10] Y.A. Drozd, A.V. Roiter, Commutative rings with finitely many integral irreducible representations, Izv. Akad. Nauk SSSR, Ser. Mat. 31(4)(1967), 783–798.
[11] N. V. Dung, D. V. Huyn, P. F. Smith and R. Wisbauer, Extending Modules Pitmann Research Notes in Math. Series, Longman Harlow, (1994)
[12] A. Ghorbani, A. Haghany, Generalized Hopfian modules, J. Algebra 255(2)(2002), 324–341.

[13] K. R. Goodearl, Incompressible critical modules. Comm. Algebra 19(1980), 1845–1851.
[14] R. Gordon and J.C. Robson, Krull Dimension, Amer. Math. Soc. Memoir No. 133, Providence R. I., 1973.
[15] V.A. Hiremath, Hopfian rings and Hopfian modules, Indian J. Pure Appl. Math. 17(7)(1986), 895–900.
[16] P.W. Keef, Sub-Bassian properties on Abelian groups, Commun. Algebra 53(4) (2025), 1723-1738.
[17] T.Y. Lam, Lectures on Modules and Rings, Graduate Texts in Math. **189** Springer-Verlag, Berlin, New York, Heidelberg, 1999.
[18] S.H. Mohamed, B.J. Müller, Continuous and Discrete Modules, London Math. Soc. Lecture Note Ser., vol. 147, Cambridge Univ. Press, 1990.
[19] L. Rowen, L. Small, Hopfian and Bassian algebras, arXiv:1711.06483.
[20] K. Varadarajan, Hopfian and co-Hopfian objects, Publ. Mat., 36(1)(1992), 293–317.

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